STEM Accessibility Test Document

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Abstract

This document collects representative STEM content for testing LaTeX-to-HTML conversion with NVDA. It includes mathematics, code, tables, charts, diagrams, captions, and text alternatives from several academic disciplines.

Contents

1 Inline and Display Mathematics

The Pythagorean relationship is $a^2 + b^2 = c^2$, and Newton's second law is $F = ma$.

A simple displayed equation is

$$E = mc^2. \tag{1}$$

2 Fractions, Roots, Exponents, and Subscripts

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

For a damped second-order system, the natural frequency and damping ratio may be written as

$$\omega_n = \sqrt{\frac{k}{m}}, \tag{3}$$

$$\zeta = \frac{c}{2\sqrt{mk}}. \tag{4}$$

3 Calculus

The derivative of a polynomial may be shown line by line:

$$f(x) = x^3 - 4x^2 + 2x - 7, \tag{5}$$

$$f'(x) = 3x^2 - 8x + 2. \tag{6}$$

The definite integral of $2x$ from zero to three is

$$\int_0^3 2x \, \mathrm{d}x = \left[x^2\right]_0^3 \tag{7}$$

$$= 3^2 - 0^2 \tag{8}$$

$$= 9. \tag{9}$$

For a multivariable function $g(x, y)$, the gradient is

$$\nabla g(x, y) = \begin{bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{bmatrix}. \tag{10}$$

4 Aligned Equations and Multi-Step Derivations

Consider the first-order differential equation

$$\dot{x}(t) + 2x(t) = 4, \qquad x(0) = 1. \tag{11}$$

Multiplying by the integrating factor e^{2t} gives

$$e^{2t}\dot{x}(t) + 2e^{2t}x(t) = 4e^{2t}, \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{2t}x(t)\right) = 4e^{2t}, \tag{13}$$

$$e^{2t}x(t) - x(0) = 2\left(e^{2t} - 1\right), \tag{14}$$

$$x(t) = 2 - e^{-2t}. \tag{15}$$

5 Vectors and Matrices

Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}. \tag{16}$$

The state equation is

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu. \tag{17}$$

6 Piecewise Functions and Cases

Piecewise definitions are important in control, optimization, and signal processing. A saturation function may be written as

$$\mathrm{sat}(u) = \begin{cases} u_{\max}, & u > u_{\max}, \\ u, & -u_{\max} \le u \le u_{\max}, \\ -u_{\max}, & u < -u_{\max}. \end{cases} \tag{18}$$

7 Greek Symbols and Control Notation

Consider the Lyapunov candidate

$$V(e, \tilde{\theta}) = \frac{1}{2}e^{\mathsf{T}}Pe + \frac{1}{2}\tilde{\theta}^{\mathsf{T}}\Gamma^{-1}\tilde{\theta}, \tag{19}$$

with the derivative condition

$$\dot{V} \le -\lambda_{\min}(Q)\|e\|^2. \tag{20}$$

8 Math Inside Lists

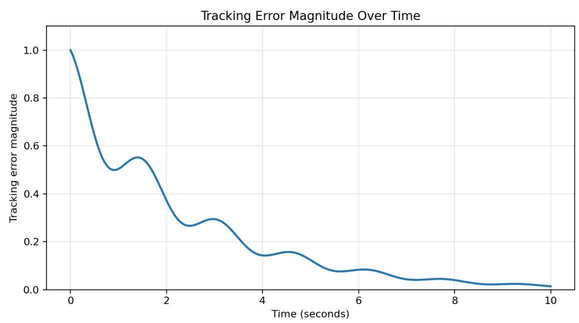
1. The sampling period is $T_s = 0.01\,\text{s}$.
2. The estimation error is $e_k = x_k - \hat{x}_k$.
3. The control command is $u_k = -Kx_k + r_k$.
4. The error magnitude is $\|e_k\|$.

9 Mathematics in a Data Table

Table 1: System parameters for a second-order example.

Parameter and meaning	**Symbol**	**Value**
Mass (object inertia)	m	$1.0\,\mathrm{kg}$
Damping coefficient (velocity resistance)	c	$0.4\,\mathrm{N\,s/m}$
Spring constant (position-restoring coefficient)	k	$2.0\,\mathrm{N/m}$
Natural frequency (undamped oscillation frequency)	ω_n	$\sqrt{k/m}$
Damping ratio (normalized damping measure)	ζ	$c/(2\sqrt{mk})$

10 Chart with a Text Equivalent



Text equivalent. From zero to ten seconds, the tracking error starts near 1.00, decreases rapidly, oscillates with progressively smaller peaks, and approaches zero. This downward trend is consistent with a stable response.

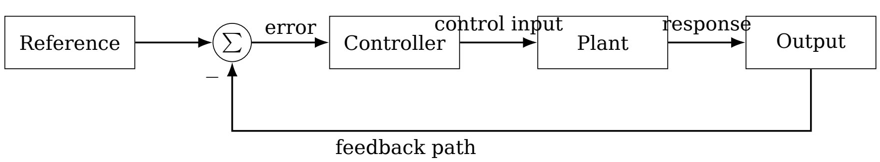
The visual shape is not the only source of information: selected numerical values are also provided in Table 2.

Figure 1: Tracking error decreases over time in a stable system.

Table 2: Selected data points from the stability chart.

Time (s)	0	2	4	6	10
Error magnitude	1.00	0.38	0.13	0.08	0.01

11 Block Diagram with a Text Equivalent



Text equivalent. The reference signal enters a summing junction. The measured output is returned through a feedback path and subtracted from the reference. The resulting error enters the controller. The controller sends a control input to the plant, and the plant produces the system output.

Figure 2: Closed-loop control diagram with reference input, controller, plant, output, and feedback.

12 Algorithm as Structured Text

1. Initialize the state estimate $\hat{\mathbf{x}}_0$, parameter estimate $\hat{\theta}_0$, and sampling period T_s.
2. At sample k, obtain the measurement y_k.
3. Compute the estimation error

$$e_k = y_k - \hat{y}_k.$$

4. Update the state estimate using the system model and correction term.
5. Update the parameter estimate according to the specified adaptive law.
6. Increase k by one and repeat until the stopping condition is met.

13 Mathematics and Statistics

13.1 Linear Equation With Explicit Steps

Solve $3x + 5 = 20$:

$$3x + 5 = 20, \tag{21}$$
$$3x = 15, \tag{22}$$
$$x = 5. \tag{23}$$

Each row performs one operation on both sides of the equality.

13.2 Quadratic Equation by Completing the Square

For $x^2 - 6x + 5 = 0$,

$$x^2 - 6x = -5, \tag{24}$$
$$x^2 - 6x + 9 = 4, \tag{25}$$
$$(x-3)^2 = 4, \tag{26}$$
$$x - 3 = \pm 2, \tag{27}$$
$$x = 1 \quad \text{or} \quad x = 5. \tag{28}$$

13.3 Derivative With Separate Left and Right Sides

Verify that

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + 3x) = 2x + 3. \tag{29}$$

Left-hand side:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + 3x) = \frac{\mathrm{d}}{\mathrm{d}x}(x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(3x) \tag{30}$$
$$= 2x + 3. \tag{31}$$

Right-hand side:

$$2x + 3. \tag{32}$$

Therefore, the independently interpreted left- and right-hand sides agree.

13.4 Matrix Multiplication

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}. \tag{33}$$

Then

$$A\mathbf{x} = \begin{bmatrix} 1(5) + 2(6) \\ 3(5) + 4(6) \end{bmatrix} \tag{34}$$
$$= \begin{bmatrix} 17 \\ 39 \end{bmatrix}. \tag{35}$$

13.5 Proof by Mathematical Induction

Prove that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \tag{36}$$

for every positive integer n.
Base case. For $n = 1$,

$$1 = \frac{1(1+1)}{2} = 1. \tag{37}$$

Inductive hypothesis. Assume

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}. \tag{38}$$

Inductive step. Then

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) \tag{39}$$
$$= (k+1)\left(\frac{k}{2} + 1\right) \tag{40}$$
$$= \frac{(k+1)(k+2)}{2}. \tag{41}$$

This is the required formula with $n = k+1$; therefore, the statement holds for all positive integers.

13.6 Separate LHS and RHS Trigonometric Derivations

Verify the identity

$$\frac{1-\cos(2x)}{\sin(2x)} = \tan(x). \tag{42}$$

Left-hand side derivation:

$$\begin{aligned}
\frac{1-\cos(2x)}{\sin(2x)} &= \frac{2\sin^2(x)}{2\sin(x)\cos(x)} \qquad (43)\\
&= \frac{\sin(x)}{\cos(x)} \qquad (44)\\
&= \tan(x). \qquad (45)
\end{aligned}$$

Right-hand side derivation:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}. \tag{46}$$

Both sides reduce to $\sin(x)/\cos(x)$, so the identity is verified wherever the original expressions are defined.

13.7 LHS Derived to a Constant RHS

For the infinite geometric series,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2. \tag{47}$$

Let

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots . \tag{48}$$

Multiply both sides by 2:

$$2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots . \tag{49}$$

Subtract the original series from the doubled series:

$$\begin{aligned}
2S - S &= \left(2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) - \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right), \qquad (50)\\
S &= 2. \qquad (51)
\end{aligned}$$

Therefore,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2. \tag{52}$$

Here the left-hand side is derived, while the right-hand side remains the constant 2.

14 Computer Science

14.1 Cross-Language Number-Analysis Program

The four listings implement the same number-analysis task in Python, R, MATLAB, and C++.

14.1.1 Python Example

Listing 1: Python palindrome and number-analysis program

```python
def normalize_number(value: int) -> int:
    """Return the nonnegative form of an integer."""
    return abs(value)


def reverse_number(value: int) -> int:
    number = normalize_number(value)
    reversed_value = 0

    while number > 0:
        digit = number % 10
        reversed_value = (reversed_value * 10) + digit
        number //= 10

    return reversed_value


def digit_sum(value: int) -> int:
    number = normalize_number(value)
    total = 0

    while number > 0:
        total += number % 10
        number //= 10

    return total


def is_palindrome(value: int) -> bool:
    number = normalize_number(value)
    return number == reverse_number(number)


def is_prime(value: int) -> bool:
    number = normalize_number(value)

    if number < 2:
        return False

    divisor = 2
    while divisor * divisor <= number:
        if number % divisor == 0:
            return False
        divisor += 1

    return True


def analyze_numbers(values: list[int]) -> list[dict[str, object]]:
    results = []

    for index, value in enumerate(values, start=1):
        result = {
            "item": index,
            "original": value,
            "reverse": reverse_number(value),
            "digit_sum": digit_sum(value),
            "palindrome": is_palindrome(value),
            "prime": is_prime(value),
        }
        results.append(result)

    return results
```

```python
def print_report(results: list[dict[str, object]]) -> None:
    print("\nNUMBER ANALYSIS REPORT")
    print("-" * 76)
    print(
        f"{'Item':>4} | {'Number':>10} | {'Reverse':>10} | "
        f"{'Digit sum':>9} | {'Palindrome':>10} | {'Prime':>5}"
    )
    print("-" * 76)

    for result in results:
        palindrome_text = "yes" if result["palindrome"] else "no"
        prime_text = "yes" if result["prime"] else "no"
        print(
            f"{result['item']:>4} | {result['original']:>10} | "
            f"{result['reverse']:>10} | {result['digit_sum']:>9} | "
            f"{palindrome_text:>10} | {prime_text:>5}"
        )


def main() -> None:
    default_data = "121, 1331, 42, 97, -44, 1001"
    prompt = f"Enter integers separated by commas [{default_data}]: "
    raw_text = input(prompt).strip()

    if raw_text == "":
        raw_text = default_data

    try:
        values = [int(part.strip()) for part in raw_text.split(",")]
    except ValueError as error:
        print(f"Input error: {error}. The default list will be used.")
        values = [121, 1331, 42, 97, -44, 1001]

    results = analyze_numbers(values)
    print_report(results)

    palindrome_count = 0; prime_count = 0
    for result in results:
        palindrome_count += int(result["palindrome"])
        prime_count += int(result["prime"])

    summary = {
        "total": len(results),
        "palindromes": palindrome_count,
        "primes": prime_count,
    }
    print(
        f"\nSummary: total={summary['total']}, "
        f"palindromes={summary['palindromes']}, "
        f"primes={summary['primes']}."
    )


if __name__ == "__main__":
    main()
```

14.1.2 R Example

The R version performs the same task so that code navigation and punctuation pronunciation can be compared across languages.

Listing 2: R palindrome and number-analysis program

```r
normalize_number <- function(value) {
  return(abs(as.integer(value)))
}

reverse_number <- function(value) {
  number <- normalize_number(value)
  reversed_value <- 0L

  while (number > 0L) {
    digit <- number %% 10L
    reversed_value <- (reversed_value * 10L) + digit
    number <- number %/% 10L
  }

  return(reversed_value)
}

digit_sum <- function(value) {
  number <- normalize_number(value)
  total <- 0L

  while (number > 0L) {
    total <- total + (number %% 10L)
    number <- number %/% 10L
  }

  return(total)
}

is_palindrome <- function(value) {
  number <- normalize_number(value)
  return(number == reverse_number(number))
}

is_prime_number <- function(value) {
  number <- normalize_number(value)

  if (number < 2L) {
    return(FALSE)
  }

  limit <- floor(sqrt(number))
  if (limit >= 2L) {
    for (divisor in 2L:limit) {
      if ((number %% divisor) == 0L) {
        return(FALSE)
      }
    }
  }

  return(TRUE)
}

analyze_numbers <- function(values) {
  rows <- lapply(seq_along(values), function(index) {
    value <- values[index]

    data.frame(
      Item = index,
      Number = value,
      Reverse = reverse_number(value),
      Digit.Sum = digit_sum(value),
      Palindrome = is_palindrome(value),
      Prime = is_prime_number(value),
      stringsAsFactors = FALSE
    )
  })

  return(do.call(rbind, rows))
}

main <- function() {
  default_data <- "121, 1331, 42, 97, -44, 1001"
  prompt_text <- paste0(
    "Enter integers separated by commas [",
    default_data,
    "]: "
  )
  raw_text <- trimws(readline(prompt = prompt_text))

  if (identical(raw_text, "")) {
    raw_text <- default_data
  }

  pieces <- trimws(unlist(strsplit(raw_text, ",")))
  values <- suppressWarnings(as.integer(pieces))

  if (length(values) == 0L || any(is.na(values))) {
    cat("Input error: the default list will be used.\n")
    values <- c(121L, 1331L, 42L, 97L, -44L, 1001L)
  }

  results <- analyze_numbers(values)
  cat("\nNUMBER ANALYSIS REPORT\n")
```

```
  print(results, row.names = FALSE)

  palindrome_count <- sum(results$Palindrome); prime_count <- sum(results$Prime)
  summary <- list(
    total = nrow(results),
    palindromes = palindrome_count,
    primes = prime_count
  )

  cat(sprintf(
    "\nSummary: total=%d, palindromes=%d, primes=%d.\n",
    summary$total,
    summary$palindromes,
    summary$primes
  ))
}

main()
```

14.1.3 MATLAB Example

The MATLAB version uses a script followed by local functions. It tests array notation, structure fields, colon-based ranges, formatted output, and function boundaries.

Listing 3: MATLAB palindrome and number-analysis program

```
% Palindrome and number-analysis program
clear; clc;

defaultData = [121, 1331, 42, 97, -44, 1001];
promptText = ['Enter integers separated by commas ', ...
              '[121, 1331, 42, 97, -44, 1001]: '];
rawText = strtrim(input(promptText, 's'));

if isempty(rawText)
    values = defaultData;
else
    pieces = split(rawText, ',');
    values = str2double(strtrim(pieces));
    values = values(:).';

    if isempty(values) || any(isnan(values))
        fprintf('Input error: the default list will be used.\n');
        values = defaultData;
    end
end

template = struct( ...
    'Item', 0, ...
    'Number', 0, ...
    'Reverse', 0, ...
    'DigitSum', 0, ...
    'Palindrome', false, ...
    'Prime', false);
results = repmat(template, 1, numel(values));

for index = 1:numel(values)
    value = values(index);
    results(index).Item = index;
    results(index).Number = value;
    results(index).Reverse = reverseNumber(value);
    results(index).DigitSum = digitSum(value);
    results(index).Palindrome = isPalindrome(value);
    results(index).Prime = isPrimeNumber(value);
end

fprintf('\nNUMBER ANALYSIS REPORT\n');
fprintf('%s\n', repmat('-', 1, 76));
fprintf('%4s | %10s | %10s | %9s | %10s | %5s\n', ...
    'Item', 'Number', 'Reverse', 'Digit sum', 'Palindrome', 'Prime');
fprintf('%s\n', repmat('-', 1, 76));

for index = 1:numel(results)
    palindromeText = yesOrNo(results(index).Palindrome);
    primeText = yesOrNo(results(index).Prime);
    fprintf('%4d | %10d | %10d | %9d | %10s | %5s\n', ...
        results(index).Item, ...
        results(index).Number, ...
        results(index).Reverse, ...
        results(index).DigitSum, ...
        palindromeText, primeText);
end

summary.total = numel(results);
summary.palindromes = sum([results.Palindrome]);
summary.primes = sum([results.Prime]);
fprintf('\nSummary: total=%d, palindromes=%d, primes=%d.\n', ...
    summary.total, summary.palindromes, summary.primes);

function output = normalizeNumber(value)
    output = abs(fix(value));
end

function output = reverseNumber(value)
    number = normalizeNumber(value);
    output = 0;

    while number > 0
        digit = mod(number, 10);
        output = (output * 10) + digit;
        number = floor(number / 10);
    end
end

function output = digitSum(value)
    number = normalizeNumber(value);
    output = 0;

    while number > 0
        output = output + mod(number, 10);
        number = floor(number / 10);
    end
end

function answer = isPalindrome(value)
    number = normalizeNumber(value);
    answer = (number == reverseNumber(number));
end

function answer = isPrimeNumber(value)
    number = normalizeNumber(value);

    if number < 2
        answer = false;
        return;
    end

    answer = true;
    limit = floor(sqrt(number));

    for divisor = 2:limit
        if mod(number, divisor) == 0
            answer = false;
            break;
        end
    end
end

function text = yesOrNo(condition)
    if condition
        text = 'yes';
    else
        text = 'no';
    end
end
```

14.1.4 C++ Example

The C++ version uses a structure, vectors, functions, exception handling, stream input, formatted output, and the conditional operator. It provides dense punctuation that is useful when evaluating screen-reader code settings.

Listing 4: C++ palindrome and number-analysis program

```
#include <cstdlib>
#include <iomanip>
#include <iostream>
#include <sstream>
#include <stdexcept>
#include <string>
#include <vector>

struct Result {
    std::size_t item;
    long long original;
    long long reversed;
    int digitSum;
    bool palindrome;
    bool prime;
};

long long normalizeNumber(long long value) {
    return std::llabs(value);
}

long long reverseNumber(long long value) {
    long long number = normalizeNumber(value);
    long long reversedValue = 0;

    while (number > 0) {
        const int digit = static_cast<int>(number % 10);
        reversedValue = (reversedValue * 10) + digit;
        number /= 10;
    }

    return reversedValue;
}

int calculateDigitSum(long long value) {
    long long number = normalizeNumber(value);
    int total = 0;

    while (number > 0) {
        total += static_cast<int>(number % 10);
        number /= 10;
    }

    return total;
}

bool isPalindrome(long long value) {
    const long long number = normalizeNumber(value);
    return number == reverseNumber(number);
}

bool isPrimeNumber(long long value) {
    const long long number = normalizeNumber(value);

    if (number < 2) {
        return false;
    }

    for (long long divisor = 2; divisor * divisor <= number; ++divisor) {
        if (number % divisor == 0) {
            return false;
        }
    }

    return true;
}

std::string trim(const std::string& text) {
    const std::string spaces = " \t\n\r";
    const std::size_t first = text.find_first_not_of(spaces);

    if (first == std::string::npos) {
        return "";
    }

    const std::size_t last = text.find_last_not_of(spaces);
    return text.substr(first, last - first + 1);
}

std::vector<long long> parseInput(const std::string& text) {
    std::vector<long long> values;
    std::stringstream stream(text);
    std::string token;

    while (std::getline(stream, token, ',')) {
        token = trim(token);
        std::size_t usedCharacters = 0;
        const long long value = std::stoll(token, &usedCharacters);

        if (usedCharacters != token.size()) {
            throw std::invalid_argument("Unexpected characters in: " + token);
        }

        values.push_back(value);
    }

    if (values.empty()) {
        throw std::invalid_argument("No integers were entered.");
    }

    return values;
}

std::vector<Result> analyzeNumbers(const std::vector<long long>& values) {
    std::vector<Result> results;

    for (std::size_t index = 0; index < values.size(); ++index) {
        const long long value = values[index];
        results.push_back(Result{
            index + 1,
            value,
            reverseNumber(value),
            calculateDigitSum(value),
            isPalindrome(value),
            isPrimeNumber(value)
        });
    }

    return results;
}

void printReport(const std::vector<Result>& results) {
    std::cout << "\nNUMBER ANALYSIS REPORT\n";
    std::cout << std::string(76, '-') << '\n';
    std::cout << std::setw(4) << "Item" << " | "
              << std::setw(10) << "Number" << " | "
              << std::setw(10) << "Reverse" << " | "
              << std::setw(9) << "Digit sum" << " | "
              << std::setw(10) << "Palindrome" << " | "
              << std::setw(5) << "Prime" << '\n';
    std::cout << std::string(76, '-') << '\n';

    for (const Result& result : results) {
        const char* palindromeText = result.palindrome ? "yes" : "no";
        const char* primeText = result.prime ? "yes" : "no";

        std::cout << std::setw(4) << result.item << " | "
                  << std::setw(10) << result.original << " | "
                  << std::setw(10) << result.reversed << " | "
                  << std::setw(9) << result.digitSum << " | "
                  << std::setw(10) << palindromeText << " | "
                  << std::setw(5) << primeText << '\n';
    }
}

int main() {
    const std::string defaultData = "121, 1331, 42, 97, -44, 1001";
    std::cout << "Enter integers separated by commas ["
              << defaultData << "]: ";

    std::string inputText;
    std::getline(std::cin, inputText);
```

Table 3: Physics measurements with different uncertainty formats.

Quantity	Measured value	Unit
Velocity	3.42 ± 0.05	$10^5\,\mathrm{m\,s^{-1}}$
Energy	$1.50^{+0.18}_{-0.12}$	keV
Temperature	295.4 ± 0.6	K

```
    inputText = trim(inputText);

    if (inputText.empty()) {
        inputText = defaultData;
    }

    std::vector<long long> values;
    try {
        values = parseInput(inputText);
    } catch (const std::exception& error) {
        std::cerr << "Input error: " << error.what()
                  << " The default list will be used.\n";
        values = {121, 1331, 42, 97, -44, 1001};
    }

    const std::vector<Result> results = analyzeNumbers(values);
    printReport(results);

    int palindromeCount = 0;
    int primeCount = 0;
    for (const Result& result : results) {
        palindromeCount += result.palindrome ? 1 : 0;
        primeCount += result.prime ? 1 : 0;
    }

    std::cout << "\nSummary: total=" << results.size()
              << ", palindromes=" << palindromeCount
              << ", primes=" << primeCount << ".\n";

    return 0;
}
```

15 Physics and Astronomy

15.1 Physics Examples

15.1.1 Measurements, Units, and Uncertainty

A measured velocity may be reported as

$$v = (3.42 \pm 0.05) \times 10^5\,\mathrm{m\,s^{-1}}.$$

An energy measurement with asymmetric uncertainty may be written as

$$E = 1.50^{+0.18}_{-0.12}\,\mathrm{keV}.$$

15.1.2 Motion with Units and Vector Notation

Consider an object with mass $m = 2\,\mathrm{kg}$ and acceleration

$$\mathbf{a} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}\ \mathrm{m\,s^{-2}}.$$

Newton's second law gives

$$\begin{aligned} \mathbf{F} &= m\mathbf{a} && (53) \\ &= 2\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} && (54) \\ &= \begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}\ \mathrm{N}. && (55) \end{aligned}$$

The magnitude of the force is

$$\begin{aligned} \|\mathbf{F}\| &= \sqrt{6^2 + (-2)^2 + 4^2} && (56) \\ &= \sqrt{56} && (57) \\ &= 2\sqrt{14}\,\mathrm{N}. && (58) \end{aligned}$$

15.1.3 Electromagnetic Field Equations

Gauss's law for the electric field is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

The Ampere–Maxwell equation is

$$\begin{aligned} \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \\ + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \end{aligned} \quad (59)$$

The Lorentz force on a particle is

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right).$$

15.1.4 Quantum-Mechanical Notation

The time-dependent SchrÃ¶dinger equation may be written as

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle.$$

Normalization requires

$$\begin{aligned} \langle \psi \mid \psi \rangle &= \int_{-\infty}^{\infty} |\psi(x)|^2\,dx && (60) \\ &= 1. && (61) \end{aligned}$$

The expected value of position is

$$\langle x \rangle = \langle \psi | \hat{x} | \psi \rangle.$$

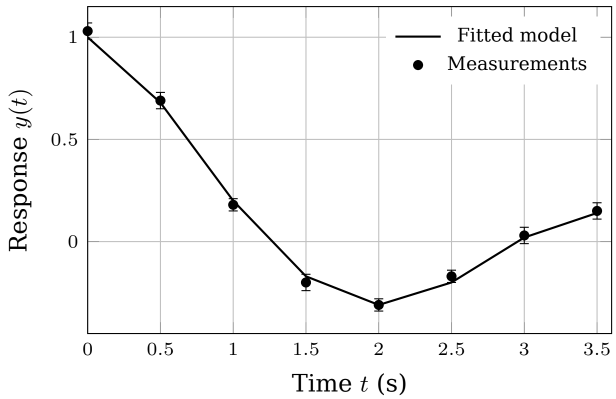


Figure 3: Measured oscillatory response with uncertainty bars and a fitted model. Filled circles represent measurements; the solid line represents the model.

Short text alternative. Measured points follow a damped oscillatory model.

Long description. Time runs from zero to 3.5 seconds. The response begins near 1.0, decreases through zero, reaches approximately -0.31 at two seconds, and then increases toward zero. Each measured point has a vertical uncertainty bar. The measured points remain close to the fitted curve.

Table 4: Data represented in the experimental-response graph.

t (s)	Measured	Uncertainty	Model
0.0	1.03	0.04	1.00
0.5	0.69	0.04	0.68
1.0	0.18	0.03	0.20
1.5	-0.20	0.04	-0.17
2.0	-0.31	0.03	-0.31
2.5	-0.17	0.03	-0.20
3.0	0.03	0.04	0.02
3.5	0.15	0.04	0.14

15.1.5 Tensor and General-Relativity Notation

Einstein's field equation is

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

An infinitesimal spacetime interval may be written as

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu.$$

15.1.6 Experimental Measurements and Fitted Model

The measured response is compared with the model

$$y(t) = e^{-0.40t} \cos(1.80t).$$

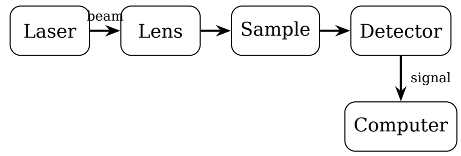


Figure 4: Simplified optical measurement apparatus. Arrows show the direction of the optical beam and recorded signal.

Short text alternative. A laser beam passes through a lens and sample before reaching a detector.

Long description. From left to right, the laser sends a beam to the lens, the lens directs the beam through the sample, and the transmitted beam reaches the detector. The detector sends an electrical signal downward to a computer.

15.1.7 Experimental-Apparatus Diagram

15.2 Astronomy Examples

15.2.1 Orbital Period and Received Flux

For two bodies with masses M and m, the orbital period of a circular orbit with radius a is

$$T = 2\pi\sqrt{\frac{a^3}{G(M+m)}}.$$

The received flux from a source with luminosity L at distance d is

$$F = \frac{L}{4\pi d^2}.$$

If the distance doubles while luminosity remains constant, then

$$F_{\text{new}} = \frac{L}{4\pi(2d)^2} \tag{62}$$
$$= \frac{L}{16\pi d^2} \tag{63}$$
$$= \frac{1}{4}F. \tag{64}$$

15.2.2 Distance Modulus and Redshift

The distance modulus is

$$m - M = 5\log_{10}\left(\frac{d}{10\,\text{pc}}\right).$$

Astronomical redshift is

$$z = \frac{\lambda_{\text{obs}} - \lambda_0}{\lambda_0}.$$

For $\lambda_{\text{obs}} = 660.0\,\text{nm}$ and $\lambda_0 = 656.3\,\text{nm}$,

$$z = \frac{660.0 - 656.3}{656.3} \tag{65}$$
$$\approx 5.64 \times 10^{-3}. \tag{66}$$

Table 5: Example celestial coordinates.

Source	Right ascension	Declination
A	12:34:56.7	−45:18:22
B	12:35:04.1	−45:17:49
C	12:35:15.8	−45:19:03

Table 6: Representative light-curve data.

Time (h)	Relative flux	Uncertainty
0	1.002	0.004
1	0.999	0.004
2	0.997	0.004
3	0.983	0.004
4	0.964	0.005
5	0.986	0.004
6	0.999	0.004
7	1.001	0.004
8	1.000	0.004

15.2.3 Celestial Coordinates

A source position may be written as

$$\alpha = 12^{\mathrm{h}}34^{\mathrm{m}}56.7^{\mathrm{s}}, \tag{67}$$
$$\delta = -45^\circ 18' 22''. \tag{68}$$

Here, α is right ascension and δ is declination.

15.2.4 Multi-Panel Astronomical Image

15.2.5 Astronomical Light Curve

15.2.6 Stellar Spectrum

15.2.7 Hertzsprung–Russell Diagram

15.2.8 Astronomical Source Catalog

16 Engineering

16.1 Electrical Circuit and Kirchhoff's Current Law

Consider a voltage divider with source voltage V_s, resistors R_1 and R_2, and output voltage V_o.

Kirchhoff's current law at the output node gives

Table 7: Representative spectrum values.

Wavelength (Å)	Normalized flux	Interpretation
6440	1.00	Continuum
6470	0.97	Continuum
6495	0.52	Absorption
6520	0.98	Continuum
6563	1.52	Emission
6580	1.02	Continuum
6620	0.99	Continuum

$$\frac{V_s - V_o}{R_1} = \frac{V_o}{R_2}, \tag{69}$$
$$R_2(V_s - V_o) = R_1 V_o, \tag{70}$$
$$R_2 V_s - R_2 V_o = R_1 V_o, \tag{71}$$
$$R_2 V_s = (R_1 + R_2) V_o, \tag{72}$$
$$V_o = \frac{R_2}{R_1 + R_2} V_s. \tag{73}$$

For $V_s = 12\,\mathrm{V}$, $R_1 = 2\,\mathrm{k}$□, and $R_2 = 1\,\mathrm{k}$□,

$$V_o = \frac{1}{2+1}(12) \tag{74}$$
$$= 4\,\mathrm{V}. \tag{75}$$

16.2 State-Space and Feedback-Control Models

A continuous-time linear system may be written as

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{76}$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t), \tag{77}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{78}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = 0. \tag{79}$$

The transfer function can be derived as

$$G(s) = C(sI - A)^{-1}B + D, \tag{80}$$
$$sI - A = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}, \tag{81}$$
$$(sI - A)^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}, \tag{82}$$
$$G(s) = \frac{1}{s^2 + 3s + 2}. \tag{83}$$

A closed-loop control system is shown in Figure 10.

The closed-loop transfer function is obtained from

$$E(s) = R(s) - H(s)Y(s), \tag{84}$$
$$Y(s) = G(s)K(s)E(s), \tag{85}$$
$$Y(s) = G(s)K(s)\left[R(s) - H(s)Y(s)\right], \tag{86}$$
$$\left[1 + G(s)K(s)H(s)\right]Y(s) = G(s)K(s)R(s), \tag{87}$$
$$\frac{Y(s)}{R(s)} = \frac{G(s)K(s)}{1 + G(s)K(s)H(s)}. \tag{88}$$

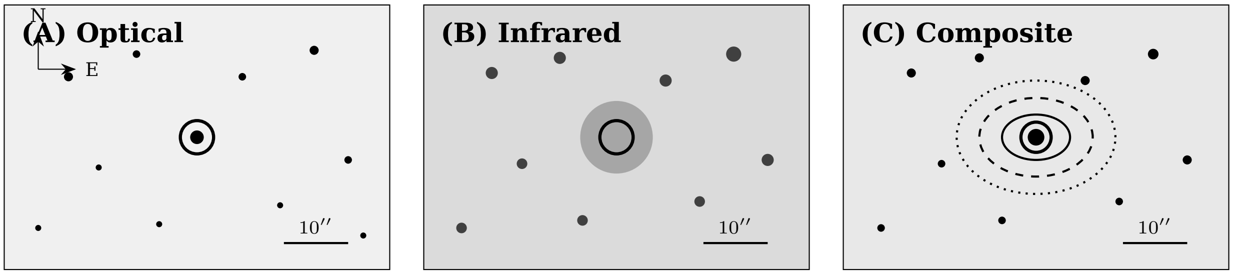


Figure 5: Controlled multi-panel astronomy image. Panel A represents an optical observation, Panel B represents an infrared observation, and Panel C combines the source positions with three contour levels. Circles identify the target; solid, dashed, and dotted contours represent increasing spatial extent.

Short text alternative. Three observations show the same astronomical field in optical, infrared, and composite representations.

Long description. All three panels show the same rectangular sky region. The target is near the center and is marked by a thick circle. Panel A contains several compact optical sources. Panel B shows the central target as a broader infrared source. Panel C overlays solid, dashed, and dotted elliptical contours around the central target. North points upward, east points right, and the scale bar represents ten arcseconds. The panel labels, contour line styles, target marker, orientation arrows, and scale bar provide information without relying only on color.

Table 8: Example astronomical source catalog containing uncertainties, limits, and unavailable values.

Source	Right ascension	Declination	g magnitude	Flux (μJy)	Redshift	Note
SRC-001	12:34:56.7	-45:18:22	18.42 ± 0.03	3.21 ± 0.14	0.125	Detected
SRC-002	12:35:04.1	-45:17:49	> 22.10	< 0.18	$\ldots$	Upper limit
SRC-003	12:35:15.8	-45:19:03	19.07 ± 0.05	2.48 ± 0.17	0.138	Detected
SRC-004	12:35:22.4	$+02$:14:08	20.31 ± 0.08	0.91 ± 0.12	0.207	Low signal

Table note. A greater-than sign in the magnitude column indicates that the source is fainter than the stated limiting magnitude. A less-than sign in the flux column indicates an upper limit. An ellipsis indicates that no redshift is available; it does not mean zero.

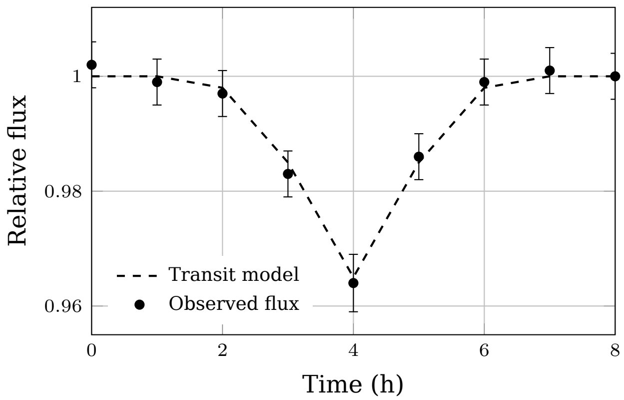


Figure 6: Example transit light curve. Filled circles with error bars represent observed flux; the dashed line represents the fitted transit model.

Short text alternative. Relative flux temporarily decreases near four hours.

Long description. Relative flux remains near 1.00 before decreasing to approximately 0.964 at four hours. It then returns to approximately 1.00. This symmetric decrease represents a possible transit. Vertical error bars show measurement uncertainty.

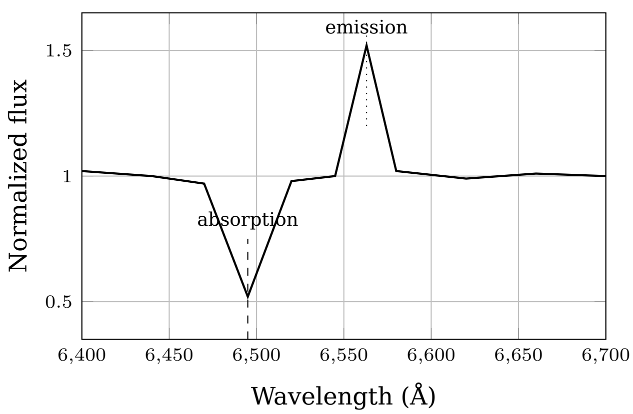


Figure 7: Example stellar spectrum containing an absorption feature near 6495 angstroms and an emission feature near 6563 angstroms.

Short text alternative. The spectrum contains one strong absorption feature and one strong emission feature.

Long description. Normalized flux is close to 1.0 across most wavelengths. It decreases to approximately 0.52 near 6495 angstroms, producing an absorption line. It increases to approximately 1.52 near 6563 angstroms, producing an emission line.

Table 9: Voltage-divider component connections.

Component	From	To	Value
Source	Ground	Input node	12 V
R_1	Input node	Output node	2 k□
R_2	Output node	Ground	1 k□

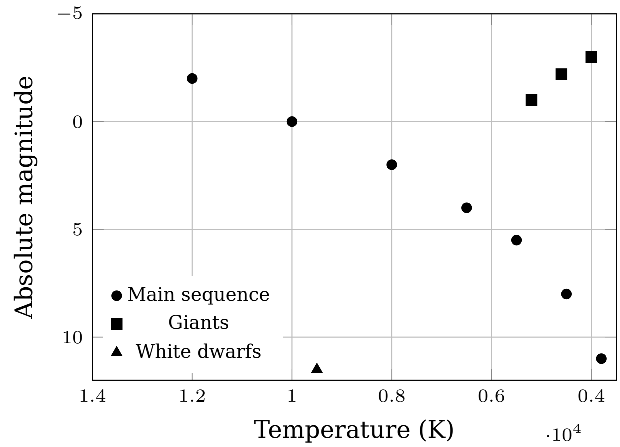


Figure 8: Controlled Hertzsprung–Russell diagram. Circles represent main-sequence stars, squares represent giants, and triangles represent white dwarfs.

Short text alternative. Three stellar groups occupy different regions of a temperature–magnitude diagram.

Long description. Temperature decreases from left to right. Absolute magnitude is also displayed in reverse order, with brighter negative magnitudes near the top. Main-sequence stars form a diagonal pattern from hot and bright at upper left to cool and faint at lower right. Giants appear near the upper-right region. White dwarfs appear near the lower-left region. Marker shape, rather than color alone, identifies each group.

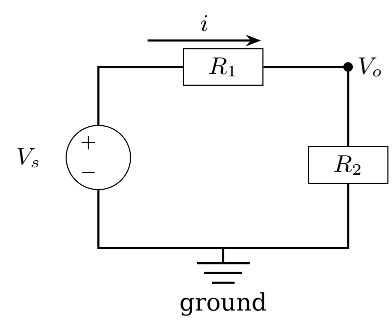


Figure 9: Voltage-divider circuit. Current i flows from the source through R_1. The output node connects R_1 to R_2, and R_2 connects the output node to ground.

Short text alternative. A voltage source and two series resistors form a voltage divider.

Long description. The positive terminal of source V_s connects to R_1. The other terminal of R_1 is the output node V_o. Resistor R_2 connects the output node to ground. The negative terminal of the source also connects to ground. Current i flows from the source through R_1 toward the output node.

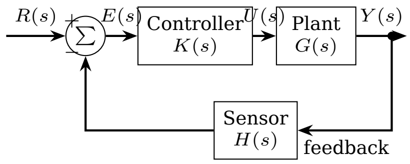


Figure 10: Closed-loop control system with negative feedback.

Short text alternative. A controller and plant form a forward path with sensor-based negative feedback.

Long description. The reference $R(s)$ enters the positive input of a summing junction. The error $E(s)$ passes through controller $K(s)$, producing control signal $U(s)$. The plant $G(s)$ produces output $Y(s)$. The output also passes through sensor $H(s)$ and returns to the negative input of the summing junction.

Table 10: Numerical values represented in the Bode plots.

ω	Magnitude (dB)	Phase (°)
0.1	−0.04	−5.7
0.2	−0.17	−11.3
0.5	−0.97	−26.6
1.0	−3.01	−45.0
2.0	−6.99	−63.4
5.0	−14.15	−78.7
10.0	−20.04	−84.3

An actuator magnitude limit $u_{\max} > 0$ may be represented by

$$\text{sat}(v) = \begin{cases} -u_{\max}, & v < -u_{\max}, \\ v, & -u_{\max} \le v \le u_{\max}, \\ u_{\max}, & v > u_{\max}. \end{cases}$$

The implemented control input is

$$u(t) = \text{sat}\left(-K\mathbf{x}(t)\right).$$

16.3 Bode Magnitude and Phase Response

For the first-order system

$$G(s) = \frac{1}{s+1},$$

the frequency response is

$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}, \tag{89}$$

$$M(\omega) = 20\log_{10}|G(j\omega)|, \tag{90}$$

$$\phi(\omega) = -\tan^{-1}(\omega). \tag{91}$$

16.4 Beam Loading and Free-Body Diagram

Consider a simply supported beam of length L with a downward point load P at its midpoint.

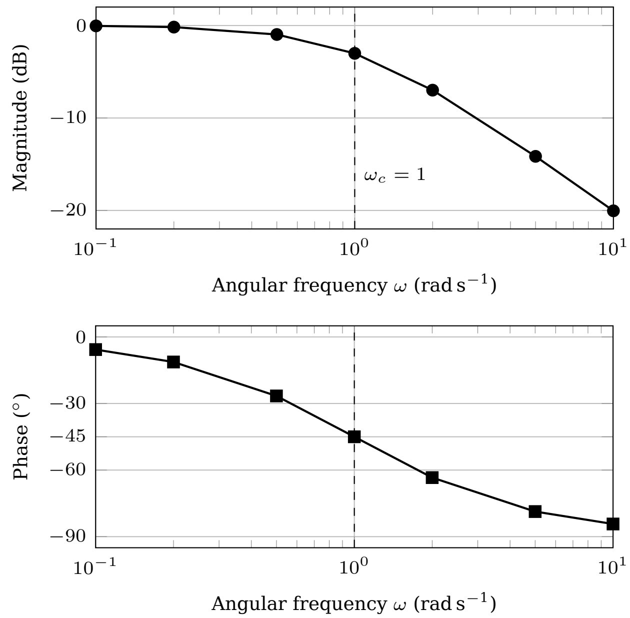


Figure 11: Bode response of $G(s) = 1/(s+1)$. The upper panel shows magnitude in decibels, and the lower panel shows phase in degrees. Both panels use a logarithmic frequency axis.

Short text alternative. Magnitude decreases and phase approaches negative 90 degrees as frequency increases.

Long description. Frequency increases logarithmically from 0.1 to 10 radians per second. Magnitude begins near zero decibels, equals approximately negative 3.01 decibels at one radian per second, and decreases to approximately negative 20.04 decibels. Phase decreases from approximately negative 5.7 degrees to negative 84.3 degrees. The cutoff frequency is one radian per second.

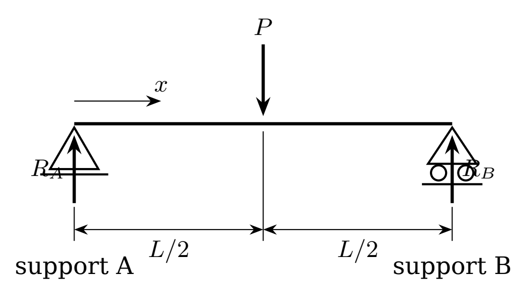


Figure 12: Free-body diagram of a simply supported beam with a midpoint load.

Short text alternative. A midpoint load acts downward on a beam with upward reactions at both ends.

Long description. A horizontal beam extends from pin support A at $x = 0$ to roller support B at $x = L$. A downward force P acts at $x = L/2$. Reactions R_A and R_B act upward at the two supports. The positive x-direction points from support A toward support B.

Force and moment equilibrium give

$$\sum F_y = R_A + R_B - P = 0, \tag{92}$$

$$\sum M_A = R_B L - P\frac{L}{2} = 0, \tag{93}$$

$$R_A = R_B = \frac{P}{2}. \tag{94}$$

The internal shear force is

$$V(x) = \begin{cases} \frac{P}{2}, & 0 < x < \frac{L}{2}, \\ -\frac{P}{2}, & \frac{L}{2} < x < L. \end{cases}$$

The bending moment is

$$M(x) = \begin{cases} \frac{Px}{2}, & 0 \le x \le \frac{L}{2}, \\ \frac{P(L-x)}{2}, & \frac{L}{2} \le x \le L. \end{cases}$$

The maximum moment occurs at the midpoint:

$$M_{\text{max}} = M\left(\frac{L}{2}\right) = \frac{PL}{4}.$$

16.5 CAD Drawing, Dimensions, and Tolerances

Figure 13 represents a rectangular plate containing one centered through-hole. All dimensions are in millimeters.

16.6 Material Stress–Strain Response

Engineering stress and strain are defined as

$$\sigma = \frac{F}{A_0}, \tag{95}$$

$$\varepsilon = \frac{\Delta L}{L_0}. \tag{96}$$

Within the linear elastic region, Young's modulus is

$$E = \frac{d\sigma}{d\varepsilon}.$$

16.7 Chemical-Process Flow and Recycle Loop

A simplified chemical process contains a mixer, reactor, separator, product stream, and recycle stream.

At steady state, the total mass balances are

$$\dot{m}_1 + \dot{m}_5 = \dot{m}_2, \tag{97}$$

$$\dot{m}_2 = \dot{m}_3, \tag{98}$$

$$\dot{m}_3 = \dot{m}_4 + \dot{m}_5. \tag{99}$$

Combining these equations gives the overall process balance

$$\dot{m}_1 = \dot{m}_4.$$

16.8 Robotic Coordinate Frames and Transformations

The homogeneous transformation from coordinate frame B to coordinate frame A is

$${}^A\mathbf{T}_B = \begin{bmatrix} {}^A\mathbf{R}_B & {}^A\mathbf{p}_B \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

A point expressed in frame B is transformed into frame A using

$${}^A\mathbf{p} = {}^A\mathbf{R}_B \, {}^B\mathbf{p} + {}^A\mathbf{p}_B.$$

For a planar rotation through angle θ,

$${}^A\mathbf{R}_B = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

16.9 Finite-Element Simulation Result

A finite-element model may approximate displacement using

$$\mathbf{Kd} = \mathbf{f},$$

where $\mathbf{K}$ is the global stiffness matrix, $\mathbf{d}$ is the nodal displacement vector, and $\mathbf{f}$ is the applied nodal-force vector.

For a fixed boundary,

$$\mathbf{d} = \mathbf{0} \qquad \text{on } \Gamma_{\text{fixed}}.$$

17 Economics and Finance

17.1 Market Equilibrium and a Per-Unit Tax

Let Q_d denote quantity demanded, Q_s quantity supplied, P_b the price paid by buyers, and P_s the price received by sellers. Consider

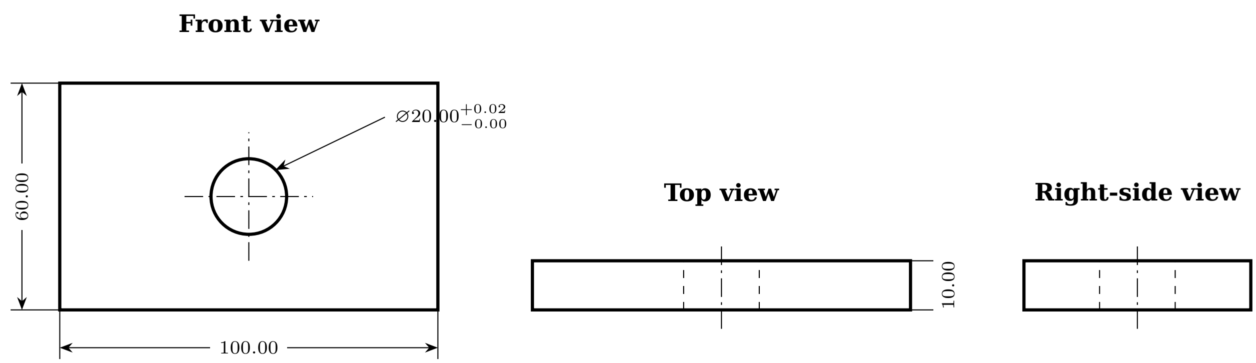


Figure 13: Orthographic drawing of a rectangular plate. The front view shows the plate length, width, hole diameter, and centerlines. The top and right-side views show plate thickness and hidden edges of the through-hole.

Short text alternative. Three orthographic views describe a rectangular plate with one centered through-hole.

Long description. The front view is a 100-millimeter by 60-millimeter rectangle. A 20-millimeter-diameter through-hole is centered 50 millimeters from either end and 30 millimeters from the top or bottom. The hole diameter has an upper tolerance of 0.02 millimeter and a lower tolerance of zero. The plate thickness is 10 millimeters. Long-short centerlines identify the hole center, while dashed hidden lines show the hole in the top and side views.

Table 11: Critical dimensions and features in the technical drawing.

Feature	Nominal value	Upper tolerance	Lower tolerance	Description
Plate length	100.00 mm	+0.10 mm	−0.10 mm	Horizontal front-view dimension
Plate width	60.00 mm	+0.10 mm	−0.10 mm	Vertical front-view dimension
Plate thickness	10.00 mm	+0.05 mm	−0.05 mm	Top and side views
Hole diameter	20.00 mm	+0.02 mm	−0.00 mm	Centered through-hole
Hole center	50.00, 30.00 mm	Not applicable	Not applicable	Measured from lower-left corner

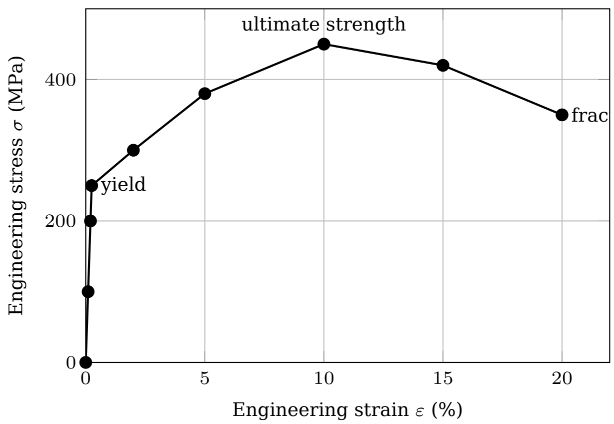


Figure 14: Controlled engineering stress–strain curve showing elastic behavior, yielding, strain hardening, ultimate tensile strength, necking, and fracture.

Short text alternative. Stress initially increases linearly, reaches a maximum, and then decreases before fracture.

Long description. Stress increases linearly from zero to approximately 250 megapascals at 0.25 percent strain. This is followed by plastic deformation and strain hardening. Ultimate tensile strength is approximately 450 megapascals at 10 percent strain. Stress then decreases during necking until fracture at approximately 20 percent strain and 350 megapascals.

Table 12: Representative stress–strain values and regions.

Strain (%)	Stress (MPa)	Region or event
0.00	0	Initial state
0.10	100	Elastic region
0.20	200	Elastic region
0.25	250	Yield point
2.00	300	Plastic region
5.00	380	Strain hardening
10.00	450	Ultimate strength
15.00	420	Necking
20.00	350	Fracture

$$Q_d = 100 - 2P_b, \tag{100}$$
$$Q_s = 20 + 2P_s. \tag{101}$$

Before a tax, $P_b = P_s = P$. The competitive equilibrium is obtained from

$$Q_d = Q_s, \tag{102}$$
$$100 - 2P = 20 + 2P, \tag{103}$$
$$80 = 4P, \tag{104}$$
$$P^* = 20, \tag{105}$$
$$Q^* = 60. \tag{106}$$

Now impose a per-unit tax of $\tau = 10$, so that

$$P_b - P_s = \tau.$$

Writing both prices as functions of the common traded quantity Q gives

$$P_b = \frac{100 - Q}{2}, \tag{107}$$
$$P_s = \frac{Q - 20}{2}. \tag{108}$$

The after-tax equilibrium follows from

$$\frac{100 - Q}{2} - \frac{Q - 20}{2} = 10, \tag{109}$$
$$\frac{120 - 2Q}{2} = 10, \tag{110}$$
$$60 - Q = 10, \tag{111}$$
$$Q_\tau = 50, \tag{112}$$
$$P_b = 25, \tag{113}$$
$$P_s = 15. \tag{114}$$

17.2 Consumer Choice and Constrained Optimization

A consumer chooses quantities x and y to maximize Cobb–Douglas utility subject to a budget constraint:

$$\max_{x,y>0} \quad U(x,y) = x^{\alpha} y^{1-\alpha}$$
$$\text{subject to} \quad p_x x + p_y y = m,$$

where p_x and p_y are prices, m is income, and $0 < \alpha < 1$. Using logarithmic utility, the Lagrangian is

$$\mathcal{L} = \alpha \ln x + (1-\alpha) \ln y + \lambda \left(m - p_x x - p_y y\right).$$

The first-order conditions are

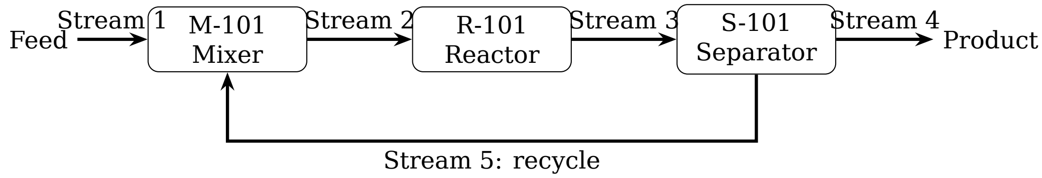


Figure 15: Chemical-process flow diagram with a recycle loop. Material flows from the feed through the mixer, reactor, and separator. Part of the separator output returns to the mixer.

Short text alternative. A feed passes through a mixer, reactor, and separator; one separator stream is recycled to the mixer.

Long description. Stream 1 carries fresh feed into mixer M-101. The mixer combines Stream 1 with recycle Stream 5. Stream 2 carries the combined material to reactor R-101. Stream 3 carries reactor output to separator S-101. The separator divides the material into product Stream 4 and recycle Stream 5. Stream 5 travels from the separator back to the mixer, creating a closed recycle loop.

Table 13: Structured description of the chemical-process streams.

Stream	From	To	Direction	Purpose
1	Fresh-feed source	Mixer M-101	Forward	Supplies new material
2	Mixer M-101	Reactor R-101	Forward	Carries combined feed
3	Reactor R-101	Separator S-101	Forward	Carries reactor output
4	Separator S-101	Product outlet	Forward	Removes final product
5	Separator S-101	Mixer M-101	Reverse/recycle	Returns unprocessed material

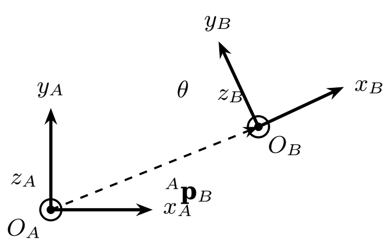


Figure 16: Two robotic coordinate frames. Frame B is translated and rotated relative to frame A. A circle with a central dot represents each positive z-axis pointing out of the page.

Short text alternative. Coordinate frame B is translated and rotated counterclockwise from frame A.

Long description. Frame A has origin O_A, with positive x_A pointing right, positive y_A pointing upward, and positive z_A pointing out of the page. Frame B has origin O_B, located above and to the right of O_A. Its x_B- and y_B-axes are rotated counterclockwise through angle θ. The vector from O_A to O_B is ${}^{A}\mathbf{p}_B$.

Table 14: Coordinate-frame notation used in the transformation.

Symbol	Meaning
${}^{A}\mathbf{T}_B$	Transform from frame B to frame A
${}^{A}\mathbf{R}_B$	Rotation of frame B relative to frame A
${}^{A}\mathbf{p}_B$	Position of origin B, expressed in frame A
${}^{B}\mathbf{p}$	Point coordinates expressed in frame B
${}^{A}\mathbf{p}$	Same point expressed in frame A

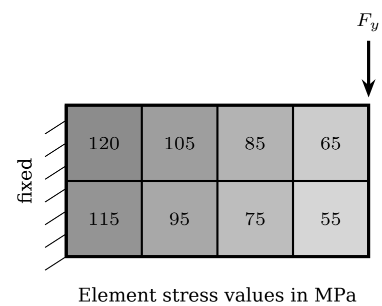


Figure 17: Controlled finite-element mesh with a fixed left boundary and a downward force at the upper-right corner. Numerical labels give representative von Mises stress values in megapascals.

Short text alternative. Stress is greatest near the fixed boundary and decreases toward the loaded right edge.

Long description. A rectangular model is divided into eight elements arranged in two rows and four columns. The entire left edge is fixed. A downward force acts at the upper-right corner. Representative stress values range from 120 megapascals in the upper-left element to 55 megapascals in the lower-right element. Darker shading corresponds to higher stress, but every element also contains a numerical value so that color or shading is not required.

Table 15: Representative finite-element results.

Region	Stress (MPa)	Displacement (mm)
Upper left	120	0.00
Lower left	115	0.00
Upper center-left	105	0.12
Lower center-left	95	0.10
Upper center-right	85	0.28
Lower center-right	75	0.24
Upper right	65	0.46
Lower right	55	0.39

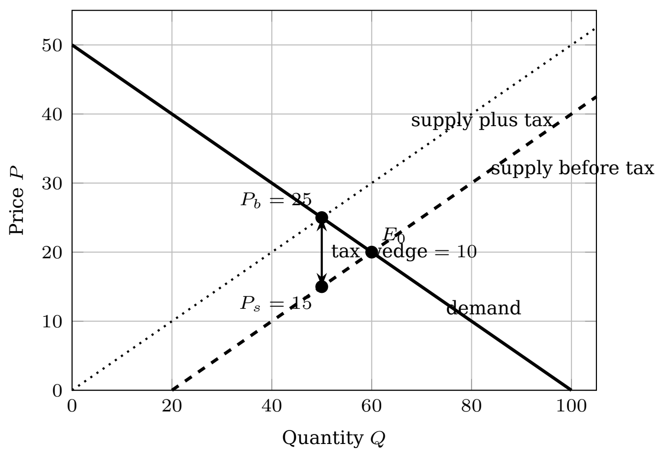


Figure 18: Supply and demand before and after a per-unit tax. The solid demand curve and dashed pre-tax supply curve meet at $Q = 60$ and $P = 20$. After the tax, trade falls to $Q = 50$; buyers pay 25, and sellers receive 15.

Short text alternative. A per-unit tax creates a ten-unit gap between buyer and seller prices and reduces quantity from 60 to 50.

Long description. Quantity is on the horizontal axis and price is on the vertical axis. Demand slopes downward. Supply before tax slopes upward and is dashed. Supply plus tax also slopes upward and is dotted. The pre-tax equilibrium is quantity 60 and price 20. The after-tax quantity is 50, with buyer price 25 and seller price 15. A vertical double-headed arrow at quantity 50 marks the ten-unit tax wedge.

Table 16: Market outcomes before and after the tax.

Outcome	Before tax	After tax
Quantity traded	60	50
Price paid by buyers	20	25
Price received by sellers	20	15
Tax per unit	0	10

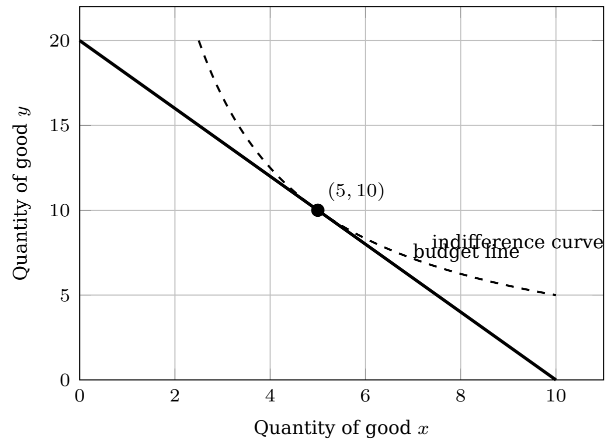


Figure 19: Consumer optimum for income 100, prices $p_x = 10$ and $p_y = 5$, and equal utility weights. The budget line is tangent to the dashed indifference curve at $(5, 10)$.

Short text alternative. The consumer's highest attainable indifference curve touches the budget line at five units of x and ten units of y.

Long description. Good x is on the horizontal axis and good y is on the vertical axis. The straight budget line runs from $y = 20$ when $x = 0$ to $x = 10$ when $y = 0$. A dashed, downward-curving indifference curve is tangent to the budget line at $x = 5$, $y = 10$. The text identifies the curve types, so line shape and location are not the only means of identification.

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\alpha}{x} - \lambda p_x = 0, \tag{115}$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{1-\alpha}{y} - \lambda p_y = 0, \tag{116}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = m - p_x x - p_y y = 0. \tag{117}$$

Dividing the first condition by the second and using the budget constraint gives

$$\frac{\alpha y}{(1-\alpha)x} = \frac{p_x}{p_y}, \tag{118}$$

$$y = \frac{(1-\alpha)p_x x}{\alpha p_y}, \tag{119}$$

$$p_x x + p_y \left[\frac{(1-\alpha)p_x x}{\alpha p_y}\right] = m, \tag{120}$$

$$\frac{p_x x}{\alpha} = m, \tag{121}$$

$$x^* = \frac{\alpha m}{p_x}, \tag{122}$$

$$y^* = \frac{(1-\alpha)m}{p_y}. \tag{123}$$

For $\alpha = 0.5$, $m = 100$, $p_x = 10$, and $p_y = 5$, the optimal bundle is $(x^*, y^*) = (5, 10)$.

The point price elasticity of demand is

$$\varepsilon_{Q,P} = \frac{\partial Q_d}{\partial P}\frac{P}{Q_d}.$$

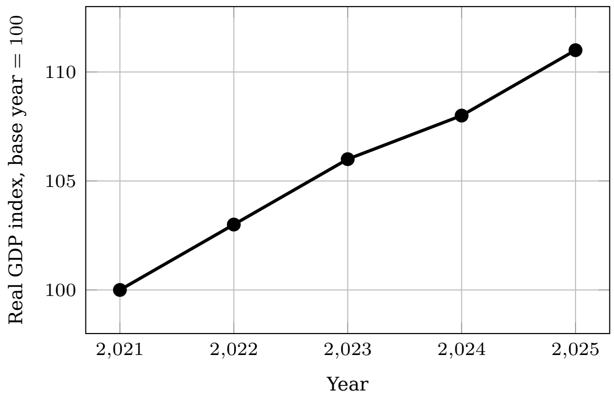


Figure 20: Illustrative real GDP index from 2021 through 2025. The index rises from 100 to 111; these controlled values are examples, not reported statistics for a particular country.

Short text alternative. The illustrative real GDP index increases in every year from 2021 to 2025.

Long description. Year is on the horizontal axis and a real GDP index with base year equal to 100 is on the vertical axis. The plotted values are 100, 103, 106, 108, and 111 for 2021 through 2025, respectively. The increase is three index points in each of the first two intervals, two points in the next interval, and three points in the final interval.

Demand is elastic when $|\varepsilon_{Q,P}| > 1$, unit elastic when $|\varepsilon_{Q,P}| = 1$, and inelastic when $|\varepsilon_{Q,P}| < 1$.

17.3 Macroeconomic Identities, Indexes, and Growth Rates

Gross domestic product in expenditure form is

$$Y = C + I + G + (X - M),$$

where C is consumption, I investment, G government purchases, X exports, and M imports.

If nominal gross domestic product and the price deflator use the same base year, real gross domestic product is

$$\text{Real GDP}_t = \frac{\text{Nominal GDP}_t}{\text{GDP deflator}_t/100}.$$

Inflation and continuously compounded output growth may be written as

$$\pi_t = 100\left(\frac{\text{CPI}_t - \text{CPI}_{t-1}}{\text{CPI}_{t-1}}\right), \tag{124}$$

$$g_t = 100\left[\ln(Y_t) - \ln(Y_{t-1})\right]. \tag{125}$$

A change from 4.25% to 4.75% is an increase of 0.50 percentage point, which is also 50 basis points:

$$4.75\% - 4.25\% = 0.50 \text{ percentage point} = 50 \text{ basis points}.$$

Table 17: Numerical values represented in the real GDP example.

Year	Nominal GDP	GDP deflator	Real GDP index
2021	100.0	100.0	100.0
2022	108.2	105.0	103.0
2023	116.6	110.0	106.0
2024	123.1	114.0	108.0
2025	130.0	117.1	111.0

Note. Values are controlled examples. Nominal GDP and the GDP deflator are scaled so the real GDP series is an index with 2021 equal to 100.

17.4 Econometric Model and Regression Results

For individual i in period t, a two-way fixed-effects model is

$$y_{it} = \beta_0 + \beta_1 x_{it} + \boldsymbol{\gamma}^T \mathbf{z}_{it} + \alpha_i + \delta_t + \varepsilon_{it},$$

where y_{it} is the outcome, x_{it} is the main explanatory variable, $\mathbf{z}_{it}$ is a vector of controls, α_i is an individual fixed effect, δ_t is a time fixed effect, and ε_{it} is the error term.

In matrix notation, the ordinary least-squares estimator is

$$\widehat{\boldsymbol{\beta}} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{y}.$$

A 95% confidence interval for coefficient β_j is

$$\widehat{\beta}_j \pm 1.96 \text{ SE}\left(\widehat{\beta}_j\right).$$

For a difference-in-differences design, let $\text{Treat}_i = 1$ identify the treatment group and $\text{Post}_t = 1$ identify the post-intervention period:

$$Y_{it} = \alpha + \beta\,\text{Treat}_i + \gamma\,\text{Post}_t + \delta\left(\text{Treat}_i \times \text{Post}_t\right) + \varepsilon_{it}.$$

Using the controlled group means in Table 18, the estimated treatment effect is

$$\widehat{\delta} = \left(\overline{Y}_{\text{treat,post}} - \overline{Y}_{\text{treat,pre}}\right) \tag{126}$$

$$- \left(\overline{Y}_{\text{control,post}} - \overline{Y}_{\text{control,pre}}\right), \tag{127}$$

$$= (70 - 50) - (55 - 45), \tag{128}$$

$$= 20 - 10, \tag{129}$$

$$= 10. \tag{130}$$

17.5 Compound Interest and Project Valuation

If principal P earns an annual nominal rate r, compounded n times per year for t years, the accumulated amount is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Table 18: Controlled group means for a difference-in-differences example.

Group	Pre-intervention	Post-intervention	Change
Control	45	55	10
Treatment	50	70	20
Treatment minus control	5	15	10

Table 19: Illustrative regression results for the difference-in-differences model.

Term	Estimate	Standard error	95% confidence interval
Treatment group	5.0	2.5	[0.1, 9.9]
Post-intervention period	10.0	2.0	[6.1, 13.9]
Treatment group × post-intervention period	10.0	3.0	[4.1, 15.9]
Constant	45.0	1.5	[42.1, 47.9]
Observations	400		
Individual fixed effects	Included		
Time fixed effects	Included		

Note. Values are controlled examples. The treatment-by-post coefficient is the difference-in-differences estimate. Standard errors and confidence intervals are stated explicitly; statistical significance is not encoded only with asterisks.

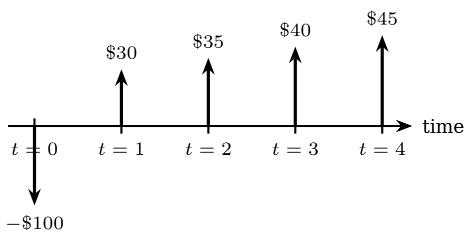


Figure 21: Cash-flow timeline for an initial investment of 100 dollars and four later inflows. The downward arrow is an outflow, and the upward arrows are inflows.

Short text alternative. An initial outflow of 100 dollars is followed by inflows of 30, 35, 40, and 45 dollars in periods one through four.

Long description. Time moves from left to right from period zero through period four. At period zero, a downward arrow represents a negative 100-dollar investment. Upward arrows represent positive cash flows of 30 dollars in period one, 35 dollars in period two, 40 dollars in period three, and 45 dollars in period four. Arrow direction is also stated in text.

For continuous compounding,

$$A = Pe^{rt}.$$

For an initial investment C_0, cash flow C_t in period t, discount rate r, and final period N, net present value is

$$\mathrm{NPV}(r) = -C_0 + \sum_{t=1}^{N} \frac{C_t}{(1+r)^t}.$$

The internal rate of return is any rate r_{IRR} satisfying

$$0 = -C_0 + \sum_{t=1}^{N} \frac{C_t}{(1+r_{\mathrm{IRR}})^t}.$$

At an annual discount rate of 8%,

$$\mathrm{NPV}(0.08) = -100 + \frac{30}{1.08} + \frac{35}{1.08^2} + \frac{40}{1.08^3} + \frac{45}{1.08^4}, \tag{131}$$

$$= \$22.61. \tag{132}$$

The decision rule is

$$\text{Decision} = \begin{cases} \text{accept the project,} & \mathrm{NPV} > 0, \\ \text{indifferent,} & \mathrm{NPV} = 0, \\ \text{reject the project,} & \mathrm{NPV} < 0. \end{cases}$$

17.6 Bond Pricing, Yield, and Duration

For a bond with coupon payment C, face value F, yield per period y, and N remaining periods, the price is

$$B_0 = \sum_{t=1}^{N} \frac{C}{(1+y)^t} + \frac{F}{(1+y)^N}.$$

For $C = \$50$, $F = \$1{,}000$, $y = 0.04$, and $N = 3$,

$$B_0 = \frac{50}{1.04} + \frac{50}{1.04^2} + \frac{1{,}050}{1.04^3}, \tag{133}$$

$$= \$1{,}027.75. \tag{134}$$

Macaulay duration and modified duration are

$$D_{\mathrm{Mac}} = \frac{\sum_{t=1}^{N} t \, \frac{\mathrm{CF}_t}{(1+y)^t}}{B_0}, \tag{135}$$

$$D_{\mathrm{mod}} = \frac{D_{\mathrm{Mac}}}{1+y}. \tag{136}$$

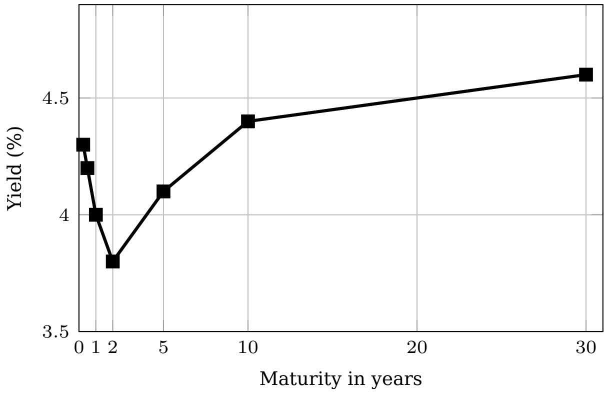


Figure 22: Illustrative yield curve. Yields decline from three-month to two-year maturities and then rise through the thirty-year maturity. Values are controlled examples, not current market quotations.

Short text alternative. The illustrative yield curve falls to a minimum at two years and then rises.

Long description. Maturity in years is on the horizontal axis and yield in percent is on the vertical axis. The plotted maturity-yield pairs are 0.25 year and 4.30%, 0.50 year and 4.20%, one year and 4.00%, two years and 3.80%, five years and 4.10%, ten years and 4.40%, and thirty years and 4.60%. The minimum illustrated yield is 3.80% at two years.

Table 20: Numerical values represented in the illustrative yield curve.

Maturity in years	Yield (%)
0.25	4.30
0.50	4.20
1.00	4.00
2.00	3.80
5.00	4.10
10.00	4.40
30.00	4.60

For a small yield change Δy, the approximate percentage price change is

$$\frac{\Delta B}{B_0} \approx -D_{\text{mod}} \Delta y.$$

17.7 Returns, Portfolio Risk, and the Efficient Frontier

For asset price P_t, prior price P_{t-1}, and distribution D_t, the simple holding-period return is

$$R_t = \frac{P_t - P_{t-1} + D_t}{P_{t-1}}.$$

For a portfolio with weight vector $\mathbf{w}$, expected-return vector $\boldsymbol{\mu}$, and covariance matrix $\boldsymbol{\Sigma}$,

$$\mathrm{E}(R_p) = \mathbf{w}^T \boldsymbol{\mu}, \tag{137}$$

$$\mathrm{Var}(R_p) = \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}, \tag{138}$$

$$\mathbf{1}^T \mathbf{w} = 1. \tag{139}$$

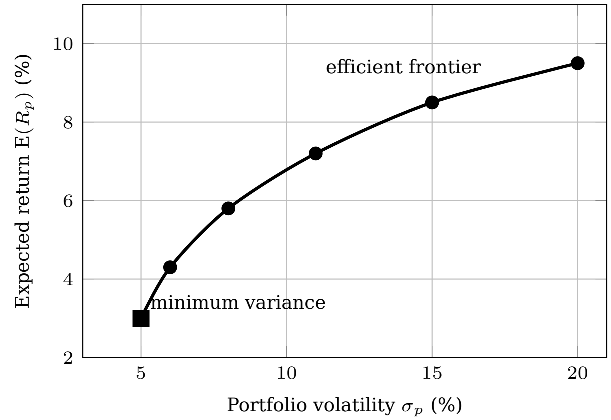


Figure 23: Controlled risk-return combinations on an illustrative efficient frontier. The square marker identifies the minimum-variance portfolio; circular markers identify other portfolios.

Short text alternative. Expected return increases as portfolio volatility increases along the illustrative efficient frontier.

Long description. Portfolio volatility in percent is on the horizontal axis and expected return in percent is on the vertical axis. Six risk-return pairs are shown: $(5, 3.0)$, $(6, 4.3)$, $(8, 5.8)$, $(11, 7.2)$, $(15, 8.5)$, and $(20, 9.5)$. The minimum-variance portfolio is the first pair and uses a square marker. The remaining pairs use circular markers. Marker shape and text, not color alone, identify the special point.

Table 21: Numerical values represented in the efficient-frontier example.

Portfolio	Volatility (%)	Expected return (%)
Minimum variance	5.0	3.0
Portfolio 2	6.0	4.3
Portfolio 3	8.0	5.8
Portfolio 4	11.0	7.2
Portfolio 5	15.0	8.5
Portfolio 6	20.0	9.5

For two assets A and B,

$$\sigma_p^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B \rho_{AB} \sigma_A \sigma_B.$$

An example covariance matrix is

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.0400 & 0.0060 \\ 0.0060 & 0.0100 \end{bmatrix}.$$

17.8 Option Payoffs and Tail-Risk Measures

For terminal underlying price S_T and strike price K, the payoffs of a European call and put are

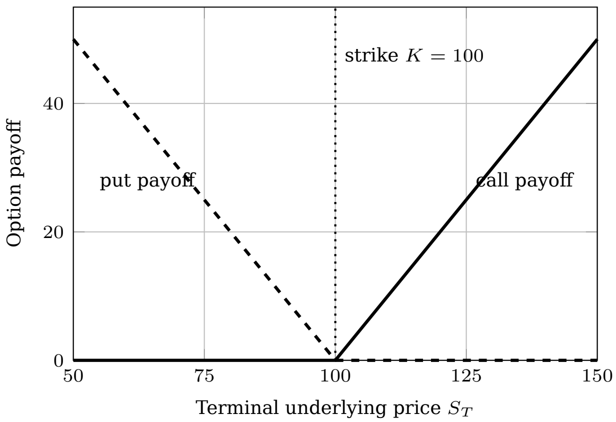


Figure 24: European call and put payoffs at expiration for strike price $K = 100$. The call is solid, the put is dashed, and the strike is dotted.

Short text alternative. Below the strike, the put has positive payoff and the call has zero payoff; above the strike, the call has positive payoff and the put has zero payoff.

Long description. Terminal underlying price ranges from 50 to 150 on the horizontal axis. Option payoff is on the vertical axis. The strike is 100. The solid call payoff is zero through 100 and then rises dollar for dollar. The dashed put payoff falls from 50 at an underlying price of 50 to zero at 100, then remains zero. A dotted vertical line marks the strike.

Table 22: Selected call and put payoffs for strike $K = 100$.

Terminal price S_T	Call payoff C_T	Put payoff P_T
80	0	20
100	0	0
120	20	0

$$C_T = \max(S_T - K, 0) = \begin{cases} 0, & S_T \le K, \\ S_T - K, & S_T > K, \end{cases} \tag{140}$$

$$P_T = \max(K - S_T, 0) = \begin{cases} K - S_T, & S_T < K, \\ 0, & S_T \ge K. \end{cases} \tag{141}$$

For loss random variable L and confidence level α,

$$\mathrm{VaR}_\alpha(L) = \inf\{\ell : \Pr(L \le \ell) \ge \alpha\}, \tag{142}$$

$$\mathrm{CVaR}_\alpha(L) = \mathrm{E}[L \mid L \ge \mathrm{VaR}_\alpha(L)]. \tag{143}$$

17.9 Financial Statements and Ratio Analysis

The balance-sheet identity is

$$\text{Assets} = \text{Liabilities} + \text{Shareholders' equity}.$$

Common ratios include

$$\text{Current ratio} = \frac{\text{Current assets}}{\text{Current liabilities}}, \tag{144}$$

$$\text{Debt-to-equity ratio} = \frac{\text{Total debt}}{\text{Shareholders' equity}}, \tag{145}$$

$$\text{Return on assets} = \frac{\text{Net income}}{\text{Average total assets}}, \tag{146}$$

$$\text{Free cash flow} = \text{Cash flow from operations} \tag{147}$$

$$- \text{Capital expenditures}. \tag{148}$$

The 2025 balance sheet satisfies

$$\text{Total assets} = \text{Total liabilities} + \text{Shareholders' equity}, \tag{149}$$

$$\$170.0 \text{ million} = \$96.0 \text{ million} + \$74.0 \text{ million}. \tag{150}$$

18 Linguistics

18.1 International Phonetic Alphabet and Transcription

Phonemic transcription uses slashes, while phonetic transcription uses square brackets. For example,

Phonemic form: /pÉªn/

Phonetic form: [pÊ°Éªn]

The superscript h in [pÊ°Éªn] marks aspiration. Common IPA symbols and modifiers include:

A minimal-pair table must keep each word, transcription, and meaning in the same row:

18.2 Acoustic Phonetics and Vowel-Space Data

The first two formant frequencies, F_1 and F_2, are often used to describe vowel quality. A controlled vowel-space plot is shown in Figure 25. The horizontal axis is reversed so that high F_2 values appear on the left, and the vertical axis is reversed so that low F_1 values appear at the top.

Voice-onset time can be defined by

$$\mathrm{VOT} = t_{\text{voicing onset}} - t_{\text{release}}.$$

For $t_{\text{release}} = 0\,\text{ms}$ and $t_{\text{voicing onset}} = 60\,\text{ms}$,

$$\mathrm{VOT} = 60\,\text{ms} - 0\,\text{ms} \tag{151}$$

$$= 60\,\text{ms}. \tag{152}$$

Table 23: Illustrative comparative financial statement.

Line item	Fiscal year 2024	Fiscal year 2025	Unit
Cash and cash equivalents	$30.0	$36.0	Millions of US dollars
Accounts receivable	$20.0	$24.0	Millions of US dollars
Property and equipment	$100.0	$110.0	Millions of US dollars
Total assets	$150.0	$170.0	Millions of US dollars
Total liabilities	$90.0	$96.0	Millions of US dollars
Shareholders' equity	$60.0	$74.0	Millions of US dollars
Net income	$15.0	$18.0	Millions of US dollars
Capital expenditures	($10.0)	($12.0)	Millions of US dollars; cash outflow

Note. Values are controlled examples. Parentheses around capital expenditures indicate negative cash flows, not optional information.

Table 24: Controlled IPA transcription examples.

Notation	Feature tested	Meaning
[ËˆfÉ”nÉ™tÉªks]	Ëˆ, É™, Éª	Primary stress and vowels
[ÊƒiË□p]	Êƒ, Ë□	Postalveolar sound and length
[sÃ£]	Ã£	Nasalized vowel
[pÊ°]	Superscript h	Aspirated consonant
[maâ□µâ□µ]	â□µâ□µ	High level tone
[maÂ¹â□µ]	Â¹â□µ	Rising tone
[maâ□µÂ¹]	â□µÂ¹	Falling tone

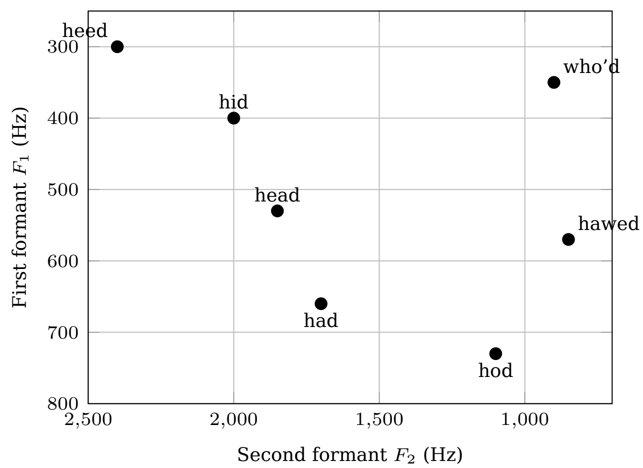


Figure 25: Controlled vowel-space plot. Both frequency axes use hertz and are reversed to approximate the orientation of the articulatory vowel space.

Short text alternative. Seven vowels are plotted by their first and second formant frequencies on two reversed axes.

Long description. The second-formant axis runs from 2500 hertz at the left to 700 hertz at the right. The first-formant axis runs from 250 hertz at the top to 800 hertz at the bottom. "Heed" is near the upper-left at $F_1 = 300$ and $F_2 = 2400$. "Who'd" is near the upper-right at $F_1 = 350$ and $F_2 = 900$. "Hod" is near the lower-right at $F_1 = 730$ and $F_2 = 1100$. Exact values for every point appear in Table 26.

Table 25: Minimal pair differing in one phoneme.

Word	Transcription	Contrasting segment
pin	/pÉªn/	/p/
bin	/bÉªn/	/b/

Table 26: Numerical values represented in the vowel-space plot.

Word	IPA	F_1 (Hz)	F_2 (Hz)
heed	/hid/	300	2400
hid	/hÉªd/	400	2000
head	/hÉ›d/	530	1850
had	/hÃ¦d/	660	1700
hod	/hÉ‘d/	730	1100
hawed	/hÉ”d/	570	850
who'd	/hud/	350	900

Table 27: Controlled Optimality Theory tableau for input /bad/.

Candidate	∗Voice-Coda	Max	Ident-Voice
[bad]	∗!		
→[bat]			∗
[ba]		∗!	

18.3 Phonological Rules, Syllables, and an Optimality Tableau

A simplified syllable may be represented by the following rules:

$$\sigma \to \text{Onset}\,\text{Rhyme}, \tag{153}$$
$$\text{Rhyme} \to \text{Nucleus}\,(\text{Coda}). \tag{154}$$

Parentheses indicate that the coda is optional. A word-final devoicing rule may be written as

$$\begin{bmatrix}-\text{sonorant}\\ +\text{voice}\end{bmatrix} \to \begin{bmatrix}-\text{voice}\end{bmatrix} \,/\, \_\_]_{\text{word}}.$$

This rule says that a voiced obstruent becomes voiceless at the end of a word. The slash introduces the conditioning environment, the underline marks the location of the changed segment, and the right bracket marks a word boundary.

The same pattern can be presented in a simplified Optimality Theory tableau:

18.4 Morphology and Interlinear Glossed Text

The word "unhelpfulness" may be segmented as

$$\underbrace{\text{un}}_{\text{prefix}} + \underbrace{\text{help}}_{\text{root}} + \underbrace{\text{ful}}_{\text{derivational suffix}} + \underbrace{\text{ness}}_{\text{nominalizing suffix}}.$$

A compact feature representation for a noun may be written as

$$\begin{bmatrix}+\text{common}\\ -\text{animate}\\ +\text{count}\end{bmatrix}.$$

Interlinear glossed text aligns a language example, a morpheme-by-morpheme gloss, and a free translation. The following controlled Turkish example illustrates plural, first-person singular possessive, and locative morphology:

The same example can be supplied in a linear fallback form:

ev-ler-im-de
house-pl-1sg.poss-loc
'in my houses'

Table 28: Interlinear glossed text with aligned morphemes.

Turkish			
ev	*-ler*	*-im*	*-de*
house	-pl	-1sg.poss	-loc
'in my houses'			

Table 29: Glossing abbreviations used in Table 28.

Abbreviation	Meaning
pl	plural
1sg	first person singular
poss	possessive
loc	locative

18.5 Judgment Marks, Phrase Structure, and Constituency

Linguistic examples may use symbols before a sentence to report different types of judgments:

A simplified grammar may use the following phrase-structure rules:

$$\text{S} \to \text{NP}\,\text{VP}, \tag{155}$$
$$\text{NP} \to \text{Det}\,\text{N}, \tag{156}$$
$$\text{VP} \to \text{V}\,\text{NP}. \tag{157}$$

The constituent structure of "The student reads the article" is shown in Figure 26.

The same hierarchy is available as bracketed text:

$$\text{S} \to [\text{NP}\,[\text{Det The}][\text{N student}]]$$
$$[\text{VP}\,[\text{V reads}][\text{NP}\,[\text{Det the}][\text{N article}]]].$$

18.6 Dependency Syntax and Corpus Annotation

A dependency analysis treats "reads" as the root. Arrows connect each head to its dependent in Figure 27.

Table 30: Common judgment marks in linguistic examples.

Mark	Example	Intended interpretation
none	The student read the article.	Judged acceptable in the stated context
∗	∗The student the article.	Morphosyntactically unacceptable
?	?The student may likely possibly leave.	Marginal or uncertain judgment
#	#The square circle slept.	Unacceptable under the intended meaning or context

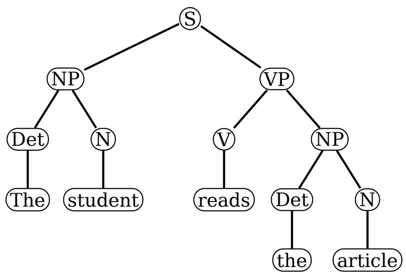


Figure 26: Constituency tree for "The student reads the article."

Short text alternative. The sentence consists of a subject noun phrase followed by a verb phrase.

Long description. The root node is S. Its first child is a noun phrase containing determiner "The" and noun "student." Its second child is a verb phrase containing verb "reads" and an object noun phrase. The object noun phrase contains determiner "the" and noun "article."

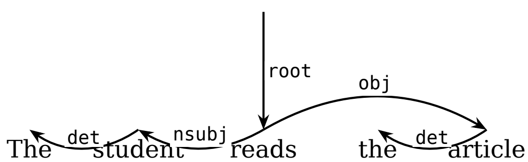


Figure 27: Dependency analysis with arrows from heads to dependents. "Reads" is the root; "student" is its nominal subject, and "article" is its object.

Short text alternative. The root "reads" governs the subject "student" and object "article"; each noun governs its determiner.

Long description. Tokens occur in the order The, student, reads, the, article. Token 3, reads, has head 0 and relation root. Token 2, student, depends on reads with relation nsubj. Token 1, The, depends on student with relation det. Token 5, article, depends on reads with relation obj. Token 4, the, depends on article with relation det. The same information appears in Table 31.

Table 31: Structured dependency annotation in a simplified CoNLL-U form.

ID	Form	Lemma	POS	Features	Head	Relation
1	The	the	DET	Definite=Def	2	det
2	student	student	NOUN	Number=Sing	3	nsubj
3	reads	read	VERB	Tense=Pres	0	root
4	the	the	DET	Definite=Def	5	det
5	article	article	NOUN	Number=Sing	3	obj

Table 32: Logical symbols in the semantic representations.

Symbol	Meaning
λx	function abstraction over x
$\forall x$	for every x
$\exists x$	there exists an x
$\wedge$	logical conjunction, "and"
$\to$	material implication, "if–then"
$>$	wider scope than
$[[\cdot]]$	semantic interpretation

18.7 Formal Semantics, Quantifiers, and Scope

Double semantic brackets denote interpretation. A controlled lexical entry is

$$[[\text{reads}]] = \lambda y.\lambda x.\ \text{Read}(x, y).$$

Generalized quantifier meanings may be written as

$$[[\text{every}]] = \lambda P.\lambda Q.\ \forall x\,(P(x) \to Q(x))\,, \tag{158}$$

$$[[\text{a}]] = \lambda P.\lambda Q.\ \exists x\,(P(x) \wedge Q(x))\,. \tag{159}$$

The sentence "Every student read a book" has at least two scope readings:

$$\begin{aligned}&\text{every} > \text{a:}\\ &\forall x\,[\text{Student}(x)\\ &\qquad \to \exists y\,(\text{Book}(y) \wedge \text{Read}(x,y))]\,,\end{aligned} \tag{160}$$

$$\begin{aligned}&\text{a} > \text{every:}\\ &\exists y\,[\text{Book}(y)\\ &\qquad \wedge \forall x\,(\text{Student}(x) \to \text{Read}(x,y))]\,.\end{aligned} \tag{161}$$

The first reading allows a potentially different book for each student. The second reading requires one book that every student read.

18.8 Pragmatics, Coreference, and Spoken-Language Transcripts

Referential indices can show possible coreference:

$$\text{Maria}_i \text{ told Lena}_j \text{ that she}_{i/j} \text{ had won.}$$

The notation i/j states that "she" may refer to Maria or Lena. It does not assert that both interpretations are equally likely in every context.

Spoken-language corpora may contain speaker labels, time codes, pauses, overlap, uncertain words, and non-speech events:

Lines 2 and 3 overlap from 00:02.1. The value (0.8) is a pause measured in seconds, double

Table 33: Controlled spoken-language transcript.

Line	Time	Speaker	Transcription
1	00:00.0--00:02.4	A	I sent the draft (0.8) yesterday.
2	00:02.1--00:03.5	B	[You sent the final one?]
3	00:02.1--00:03.7	A	[The revised one,] yes.
4	00:03.8--00:05.0	B	((laughs)) I missed that.
5	00:05.1--00:06.2	A	It was in the (shared) folder.

Table 34: Time-aligned multimodal annotation tiers.

Time interval	Dominant hand	Non-dominant hand	Nonmanual behavior	Free translation
0.0--0.8 s	index-2	neutral	brows raised	'you'
0.8--1.6 s	buy	neutral	brows raised	'buy'
1.6--2.4 s	what	neutral	brows raised	'what'

parentheses mark a non-speech event, and parentheses around "shared" mark uncertain transcription.

18.9 Signed-Language and Multimodal Annotation

Signed and multimodal data are often represented on synchronized tiers. The following notation is schematic and is not intended as a grammatical claim about a particular signed language:

raised brows across the full utterance
YOU BUY WHAT .

18.10 Historical Linguistics and Language-Family Relationships

Historical notation uses several symbols whose meanings depend on context. In the following constructed illustration,

$$^{*}\text{p} > \text{f} \,/\, \#\,\_\_,$$

the superscript asterisk marks a reconstructed form, > means "changes into," the slash introduces the environment, the number sign marks a word boundary, and the underline marks the position of the changing sound.

Table 35: Constructed sound correspondences for three daughter varieties.

Reconstruction	Variety A	Variety B	Variety C
*pata	pata	fata	hata
*pilu	pilu	filu	hilu
*puna	puna	funa	huna

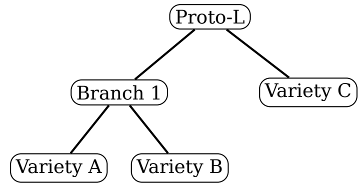


Figure 28: Constructed language-family tree. Proto-L divides into Branch 1 and Variety C; Branch 1 later divides into Varieties A and B.

Short text alternative. Proto-L has two immediate descendants: Branch 1 and Variety C.

Long description. The root is Proto-L. Its first child is Branch 1, whose children are Variety A and Variety B. The root's second child is Variety C. The vertical arrangement shows ancestry, not geographic position or chronological scale.

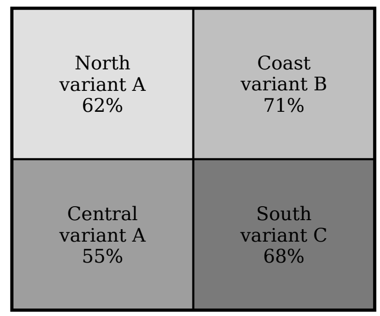


Figure 29: Schematic dialect survey with four labeled regions. Shading is redundant because each region also contains its name, leading variant, and percentage.

Short text alternative. Variant A leads in North and Central, variant B leads in Coast, and variant C leads in South.

Long description. The schematic has four rectangular regions. North is upper-left and reports variant A at 62 percent. Coast is upper-right and reports variant B at 71 percent. Central is lower-left and reports variant A at 55 percent. South is lower-right and reports variant C at 68 percent. These positions are only a controlled accessibility test and are not real geographic boundaries.

18.11 Geographic Variation and a Schematic Dialect Map

Figure 29 is a schematic survey map rather than a real geographic boundary map. Pattern and text labels identify four regions.

18.12 Corpus Frequencies and Sociolinguistic Models

A normalized corpus frequency per million words is

$$\text{Frequency}_{\text{pmw}} = \frac{\text{target-token count}}{\text{total corpus tokens}} \times 10^{6}.$$

For 420 target tokens in a corpus of 2,000,000 tokens,

Table 36: Data represented in the schematic dialect map.

Region	Leading variant	Share (%)
North	A	62
Coast	B	71
Central	A	55
South	C	68

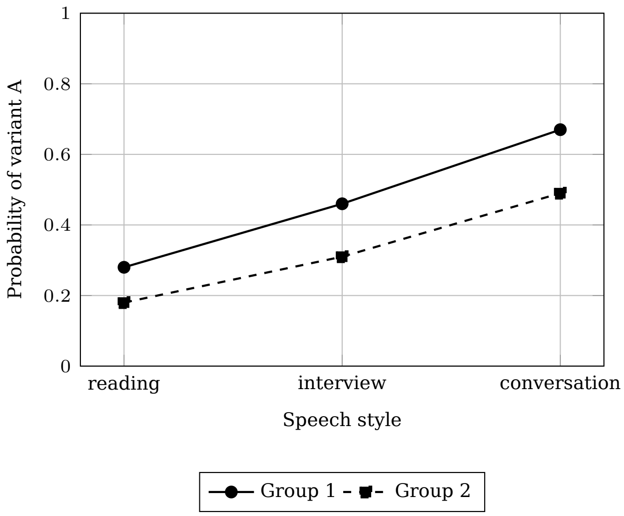


Figure 30: Controlled sociolinguistic probabilities by group and speech style. Group 1 is a solid line with circular markers; Group 2 is a dashed line with square markers.

Short text alternative. The probability of variant A rises from reading to conversation in both groups, and Group 1 is higher at every style.

Long description. The horizontal axis lists reading, interview, and conversation in that order. The vertical axis ranges from zero to one. Group 1 probabilities are 0.28, 0.46, and 0.67. Group 2 probabilities are 0.18, 0.31, and 0.49. Exact values appear in Table 37.

$$\text{Frequency}_{\text{pmw}} = \frac{420}{2{,}000{,}000} \times 10^6 \tag{162}$$

$$= 210. \tag{163}$$

The normalized frequency is 210 occurrences per million words.

A mixed-effects logistic model for a binary linguistic outcome may be written as

$$\text{logit}\,[\Pr(Y_{ij} = 1)] = \beta_0 + \beta_1\,\text{Group}_i + \beta_2\,\text{Style}_j + u_i + v_j,$$

where i indexes speakers, j indexes lexical items, β_0 is the intercept, β_1 and β_2 are fixed effects, u_i is a speaker-specific random intercept, and v_j is an item-specific random intercept.

Table 37: Numerical values in the sociolinguistic plot.

Style	Group 1	Group 2
Reading	0.28	0.18
Interview	0.46	0.31
Conversation	0.67	0.49

19 Philosophy and Logic

19.1 Logical Symbols, Scope, and Use–Mention Distinctions

Let P, Q, and R be propositions. Parentheses explicitly mark the scope of each connective:

$$\neg P \vee (Q \wedge R).$$

This formula means that either P is false, or both Q and R are true. It must not be read as

$$(\neg P \vee Q) \wedge R,$$

which has a different grouping and can have a different truth value.

Table 38: Core logical symbols and their intended readings.

Symbol	Typical reading	Structural point to preserve
$\neg P$	not P	Negation applies to P
$P \wedge Q$	P and Q	Conjunction has two operands
$P \vee Q$	P or Q	Inclusive disjunction unless stated otherwise
$P \to Q$	if P, then Q	Antecedent precedes consequent
$P \leftrightarrow Q$	P if and only if Q	Biconditional, not implication in one direction
$\forall x\, F(x)$	every x is F	Universal quantifier binds x
$\exists x\, F(x)$	some x is F	Existential quantifier binds x
$\Gamma \vdash \varphi$	φ is derivable from Γ	Syntactic consequence
$\Gamma \models \varphi$	φ is true in every model of Γ	Semantic consequence
$\varphi \equiv \psi$	φ and ψ are equivalent	Equivalence at the stated level

Philosophical writing also distinguishes using an expression from mentioning that expression:

$$\text{Boston is a city} \quad \text{uses the name Boston}, \tag{164}$$

$$\text{“Boston” has six letters} \quad \text{mentions the name Boston}. \tag{165}$$

A T-schema instance makes the distinction explicit:

$$\text{“Snow is white” is true} \quad \leftrightarrow \quad \text{snow is white.}$$

19.2 Propositional Logic and Complete Truth Tables

The following two tables test negation and all common binary truth-functional connectives in a controlled example.

Table 39: Negation, conjunction, and inclusive disjunction.

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$
True	True	False	True	True
True	False	False	False	True
False	True	True	False	True
False	False	True	False	False

Table 40: Implication, biconditional, and exclusive disjunction. The symbol $\oplus$ denotes exclusive disjunction.

P	Q	$P \to Q$	$P \leftrightarrow Q$	$P \oplus Q$
True	True	True	True	False
True	False	False	False	True
False	True	True	False	True
False	False	True	True	False

De Morgan's law states

$$\neg(P \wedge Q) \equiv (\neg P \vee \neg Q).$$

Modus ponens can be represented as an argument and as a single conditional:

$$P \to Q, \quad P \therefore Q, \tag{166}$$

$$\big[(P \to Q) \wedge P\big] \to Q \quad \text{is a tautology.} \tag{167}$$

Table 41: Classification of representative formulas.

Formula	Classification
$P \vee \neg P$	Tautology: true on every valuation
$P \wedge \neg P$	Contradiction: false on every valuation
$P \to Q$	Contingency: true on some valuations and false on others

19.3 Natural Deduction and the Scope of a Subproof

The following derivation proves $P \to R$ from $P \to Q$ and $Q \to R$. Lines 3 through 5 form a subproof whose temporary assumption is discharged on line 6.

Table 42: Line-by-line natural-deduction proof.

Line	Scope	Formula	Justification
1	Main	$P \to Q$	Premise
2	Main	$Q \to R$	Premise
3	Subproof	P	Assumption
4	Subproof	Q	$\to$ elimination, 1 and 3
5	Subproof	R	$\to$ elimination, 2 and 4
6	Main	$P \to R$	$\to$ introduction, 3–5

Common inference-rule layouts include

$$\frac{P \qquad P \to Q}{Q} \quad (\to E), \tag{168}$$

$$\frac{P \qquad Q}{P \wedge Q} \quad (\wedge I), \tag{169}$$

$$\frac{\begin{array}{c}[\neg P] \\ \vdots \\ \bot\end{array}}{P} \quad (\text{RAA}). \tag{170}$$

Here, $\bot$ denotes contradiction, I denotes introduction, E denotes elimination, and RAA denotes reductio ad absurdum.

19.4 Predicate Logic, Quantifier Order, and Identity

Consider the premises

$$\forall x\, \big(\text{Human}(x) \to \text{Mortal}(x)\big), \tag{171}$$

$$\text{Human}(\text{Socrates}). \tag{172}$$

The conclusion follows by universal instantiation and modus ponens:

$$\forall x\, \big(\text{Human}(x) \to \text{Mortal}(x)\big) \quad \text{Premise}, \tag{173}$$

$$\text{Human}(\text{Socrates}) \quad \text{Premise}, \tag{174}$$

$$\text{Human}(\text{Socrates}) \to \text{Mortal}(\text{Socrates}) \quad \forall E, \tag{175}$$

$$\text{Mortal}(\text{Socrates}) \quad \to E. \tag{176}$$

Quantifier order changes meaning:

$$\forall x\, \exists y\, \text{Admires}(x, y) \quad \text{Everyone admires someone}, \tag{177}$$

$$\exists y\, \forall x\, \text{Admires}(x, y) \quad \text{Someone is admired by everyone}. \tag{178}$$

The two quantifier-negation laws are

$$\neg\forall x\, F(x) \equiv \exists x\, \neg F(x), \tag{179}$$

$$\neg\exists x\, F(x) \equiv \forall x\, \neg F(x). \tag{180}$$

Existence and uniqueness may be written without introducing a special uniqueness quantifier:

$$\exists x\, \left[F(x) \wedge \forall y\, \big(F(y) \to y = x\big)\right].$$

This formula says that exactly one object is F.

Table 43: Quantifier-scope contrasts.

Formula	Controlled reading
$\forall x\,(F(x) \to G(x))$	Every F is a G
$\exists x\,(F(x) \wedge G(x))$	Some F is a G
$\forall x\, \exists y\, R(x, y)$	Each object bears R to at least one object
$\exists y\, \forall x\, R(x, y)$	One object is borne R by every object
$\exists x\, [F(x) \wedge \forall y(F(y) \to y = x)]$	Exactly one object is F

19.5 Categorical Logic and an Euler Diagram

Consider the categorical argument:

1. All poets are writers.
2. No writers are machines.
3. Therefore, no poets are machines.

Let P, W, and M denote the sets of poets, writers, and machines. The premises and conclusion are

$$P \subseteq W, \qquad W \cap M = \varnothing, \qquad \therefore P \cap M = \varnothing.$$

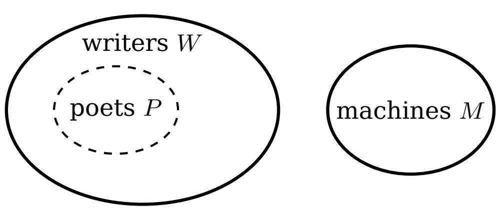


Figure 31: Euler diagram for the categorical argument. The dashed poets region is entirely inside the writers region, and the machines region is disjoint from both.

Short text alternative. Poets are a subset of writers, and machines do not overlap writers.

Long description. A large solid ellipse represents all writers. A smaller dashed ellipse representing poets lies completely inside the writers ellipse. A separate solid ellipse representing machines lies to the right and does not overlap the writers or poets ellipses. Therefore, the poets and machines regions are disjoint.

Table 44: Structured equivalent of the Euler diagram.

Region 1	Relation	Region 2
Poets P	is a subset of	Writers W
Writers W	is disjoint from	Machines M
Poets P	is disjoint from	Machines M

19.6 Sets, Relations, Functions, and a Relation Matrix

Let

$$A = \{a, b, c\}, \qquad \mathcal{R} = \{(a, b), (a, c), (b, c)\}.$$

Using the order a, b, c for both rows and columns, the adjacency matrix of $\mathcal{R}$ is

$$M_{\mathcal{R}} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The relation is transitive because

$$\forall x\, \forall y\, \forall z\, \left[\big(x\mathcal{R}y \wedge y\mathcal{R}z\big) \to x\mathcal{R}z\right].$$

A function and its composition may be written as

$$f\colon A \to B, \tag{181}$$

$$g\colon B \to C, \tag{182}$$

$$(g \circ f)(a) = g\big(f(a)\big). \tag{183}$$

Table 45: Row-by-row equivalent of the relation matrix.

From\to	a	b	c
a	0	1	1
b	0	0	1
c	0	0	0

19.7 Formal Semantics, Models, and Metalogical Consequence

A first-order model may be represented by

$$\mathcal{M} = \langle D, I \rangle,$$

where D is a nonempty domain and I is an interpretation function. Truth relative to a model $\mathcal{M}$, variable assignment s, and formula φ is written

$$\mathcal{M}, s \models \varphi.$$

Semantic and syntactic consequence must remain distinct:

$$\Gamma \models \varphi \quad \text{every model of } \Gamma \text{ satisfies } \varphi, \tag{184}$$

$$\Gamma \vdash \varphi \quad \text{there is a formal derivation of } \varphi \text{ from } \Gamma. \tag{185}$$

For a sound and complete proof system,

$$\Gamma \vdash \varphi \Longrightarrow \Gamma \models \varphi \quad \text{(soundness)}, \tag{186}$$

$$\Gamma \models \varphi \Longrightarrow \Gamma \vdash \varphi \quad \text{(completeness)}. \tag{187}$$

The satisfaction clause for conjunction is

$$\mathcal{M}, s \models (\varphi \wedge \psi) \quad \text{if and only if} \quad \mathcal{M}, s \models \varphi \text{ and } \mathcal{M}, s \models \psi.$$

Table 46: Levels of logical analysis.

Level	Example	Meaning
Object language	$P \to Q$	Formula inside the formal system
Proof theory	$\Gamma \vdash \varphi$	Derivability by formal rules
Semantics	$\mathcal{M}, s \models \varphi$	Truth in a model under an assignment
Metatheory	Soundness and completeness	Claims about relations between proof and semantics

19.8 Sequents, Proof Trees, and a Semantic Tableau

A sequent places assumptions to the left of the turnstile and conclusions to the right:

$$P,\ P \to Q \vdash Q.$$

The same inference can be displayed as a proof tree:

$$\frac{P \qquad P \to Q}{Q}\ (\to E).$$

To test whether

$$P \vee Q, \quad \neg P, \quad \neg Q$$

is satisfiable, a semantic tableau branches on $P \vee Q$.

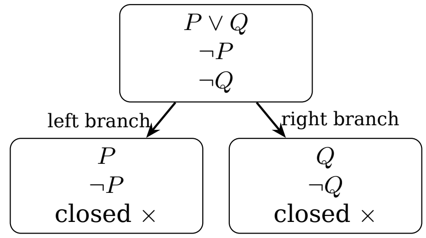


Figure 32: Closed semantic tableau. The disjunction creates two branches; the left contains P and $\neg P$, and the right contains Q and $\neg Q$.

Short text alternative. Both branches close because each contains a proposition and its negation.

Long description. The root lists $P \vee Q$, not P, and not Q. Expanding the disjunction creates a left branch containing P and a right branch containing Q. The left branch closes because it contains both P and not P. The right branch closes because it contains both Q and not Q. Since every branch closes, the original set is unsatisfiable.

Table 47: Structured description of the tableau branches.

Branch	Added formula	Conflict	Status
Left	P	$P, \neg P$	Closed
Right	Q	$Q, \neg Q$	Closed

19.9 Modal Logic and a Possible-World Model

A Kripke model is a triple

$$\mathcal{K} = \langle W, \mathcal{R}, V \rangle,$$

where W is a nonempty set of worlds, $\mathcal{R}$ is an accessibility relation, and V is a valuation.

$$\mathcal{K}, w \models \Box\varphi \quad \text{if and only if} \quad \forall v \left(w\mathcal{R}v \to \mathcal{K}, v \models \varphi \right), \tag{188}$$

$$\mathcal{K}, w \models \Diamond\varphi \quad \text{if and only if} \quad \exists v \left(w\mathcal{R}v \wedge \mathcal{K}, v \models \varphi \right). \tag{189}$$

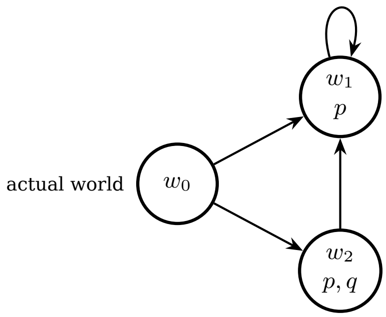


Figure 33: Possible-world model with actual world w_0. World w_0 accesses w_1 and w_2; w_1 accesses itself; and w_2 accesses w_1. Proposition p is true at w_1 and w_2, while q is true only at w_2.

Short text alternative. The actual world accesses two worlds where p is true; q is true in only one of them.

Long description. There are three worlds. From w_0, arrows lead to w_1 and w_2. World w_1 has a loop to itself. An arrow from w_2 leads to w_1. The valuation at w_1 contains p. The valuation at w_2 contains both p and q. No proposition is marked true at w_0. Consequently, $\Box p$ and $\Diamond q$ are true at w_0, but $\Box q$ is false there.

Table 48: Structured equivalent of the possible-world diagram.

World	Accessible worlds	True atoms	Result at world
w_0	w_1, w_2	none	$\Box p, \Diamond q, \neg\Box q$
w_1	w_1	p	$\Box p, \neg q$
w_2	w_1	p, q	$q, \Box p$

19.10 Modalities Used in Philosophical Logic

The same formal architecture is used for several philosophically important kinds of modality.

Table 49: Representative operators used in philosophical logic.

System	Formula	Controlled reading
Alethic	$\Box P,\ \Diamond P$	Necessarily P; possibly P
Epistemic	$K_a P$	Agent a knows that P
Doxastic	$B_a P$	Agent a believes that P
Deontic	$O(P)$, Perm(P), $F(P)$	P is obligatory, permitted, or forbidden
Temporal	GP, FP, HP, PP	Always in the future, sometime in the future, always in the past, or sometime in the past
Counterfactual	$P \Box\!\!\rightarrow Q$	If P were the case, Q would be the case
Dynamic epistemic	$[!P]K_a Q$	After the public announcement of P, agent a knows Q

Modal and quantifier scope can create a de re/de dicto contrast:

$\exists x\, \Box F(x)$ Some particular object is necessarily F, (190)

$\Box \exists x\, F(x)$ It is necessary that some object is F. (191)

One standard duality is

$$\Diamond P \equiv \neg\Box\neg P.$$

19.11 Many-Valued, Fuzzy, and Paraconsistent Logic

In a three-valued system, let truth degrees be

$$\left\{0, \frac{1}{2}, 1\right\},$$

where 0 is false, $1/2$ is indeterminate, and 1 is true. One controlled set of operations is

$$v(\neg P) = 1 - v(P), \quad (192)$$
$$v(P \wedge Q) = \min\{v(P), v(Q)\}, \quad (193)$$
$$v(P \vee Q) = \max\{v(P), v(Q)\}. \quad (194)$$

Table 50: Three-valued negation, conjunction, and disjunction.

$v(P)$	$v(Q)$	$v(\neg P)$	$v(P \wedge Q)$	$v(P \vee Q)$
0	0	1	0	0
0	$\frac{1}{2}$	1	0	$\frac{1}{2}$
0	1	1	0	1
$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1
1	0	0	0	1
1	$\frac{1}{2}$	0	$\frac{1}{2}$	1
1	1	0	1	1

A fuzzy-membership example is

$$\mu_{\text{Tall}}(\text{Alex}) = 0.7.$$

This assigns a degree of membership and does not mean that the proposition is true with probability 0.7.

Some paraconsistent systems distinguish four informational states:

Table 51: Four informational states used in a paraconsistent presentation.

Code	Evidence for P	Evidence for $\neg P$
T	Present	Absent
F	Absent	Present
B	Present	Present
N	Absent	Absent

19.12 Bayesian Epistemology and Rational Choice

Bayesian confirmation uses conditional probability:

$$\Pr(H \mid E) = \frac{\Pr(E \mid H)\Pr(H)}{\Pr(E)}.$$

If

$$\Pr(H) = 0.20, \quad (195)$$
$$\Pr(E \mid H) = 0.80, \quad (196)$$
$$\Pr(E \mid \neg H) = 0.10, \quad (197)$$

then

$$\Pr(E) = \Pr(E \mid H)\Pr(H) + \Pr(E \mid \neg H)\Pr(\neg H), \quad (198)$$
$$= (0.80)(0.20) + (0.10)(0.80), \quad (199)$$
$$= 0.24, \quad (200)$$
$$\Pr(H \mid E) = \frac{(0.80)(0.20)}{0.24}, \quad (201)$$
$$= \frac{2}{3}. \quad (202)$$

For actions a, states s, utility $u(a,s)$, and evidence E, expected utility is

$$\mathrm{EU}(a \mid E) = \sum_{s \in S} \Pr(s \mid E)\, u(a,s).$$

Table 52: Controlled decision matrix with utilities.

Action	State s_1	State s_2
a_1	10	−4
a_2	6	2

19.13 Argument Maps, Objections, and Replies

Philosophical arguments often contain supporting reasons, objections, and replies rather than a single linear proof.

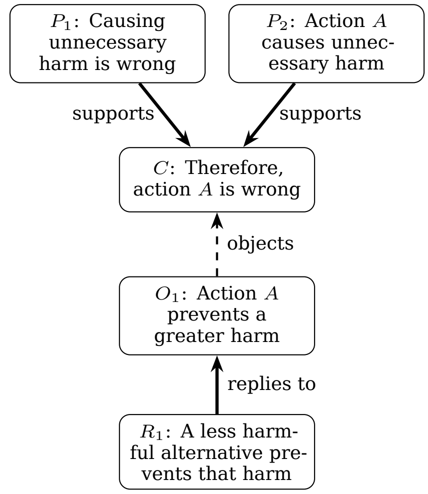


Figure 34: Argument map containing two supporting premises, a conclusion, an objection, and a reply. Solid arrows indicate support or reply; the dashed arrow indicates an objection.

Short text alternative. Two premises support the conclusion that action A is wrong; an objection appeals to preventing greater harm, and a reply identifies a less harmful alternative.

Long description. Premise P_1, "Causing unnecessary harm is wrong," and premise P_2, "Action A causes unnecessary harm," each have a solid support arrow pointing to conclusion C, "Action A is wrong." Objection O_1, "Action A prevents a greater harm," has a dashed attack arrow pointing to conclusion C. Reply R_1, "A less harmful alternative prevents that harm," has a solid reply arrow pointing to objection O_1.

Table 53: Structured equivalent of the argument map.

Identifier and role	Statement	Relation
P_1, premise	Causing unnecessary harm is wrong	Supports C
P_2, premise	Action A causes unnecessary harm	Supports C
C, conclusion	Action A is wrong	Supported by P_1, P_2
O_1, objection	Action A prevents a greater harm	Attacks C
R_1, reply	A less harmful alternative prevents that harm	Replies to O_1

19.14 Self-Reference, Quotation, and Defined Predicates

Quotation devices are especially important in discussions of truth and self-reference. Let L name the sentence that says of itself that it is not true. A compact representation is

$$L \leftrightarrow \neg \mathrm{True}(\ulcorner L \urcorner).$$

Here, $\ulcorner L \urcorner$ denotes a name or quotation of the expression L; it is not another assertion of L.

A proposed analysis of knowledge might be displayed as a definition:

$$K_aP := \big(P \wedge B_aP \wedge J_aP\big),$$

where B_aP means that agent a believes P, and J_aP means that agent a is justified in believing P. The displayed formula records a proposed analysis; it does not establish that the analysis is correct.

Table 54: Object-language and metalanguage distinctions.

Item	Level	Function
L	Object-language sentence	Makes or represents an assertion
$\ulcorner L \urcorner$	Metalanguage name	Refers to the expression L
True($\ulcorner L \urcorner$)	Semantic predication	Attributes truth to the named sentence
$:=$	Definitional notation	Introduces a proposed definition